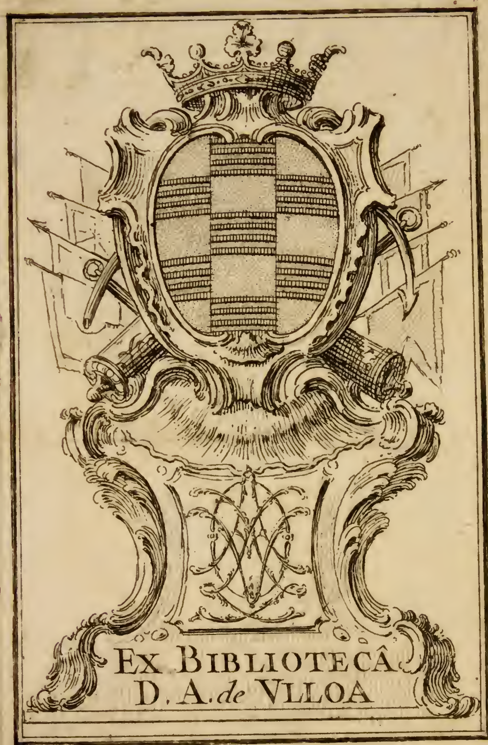
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TREATISE

OF THE

METHOD of FLUXIONS

AND

INFINITE SERIES,

With its Application to the Geometry of CURVE LINES.

By Sir ISAAC NEWTON, Kt.

Tranſlated from the *Latin* Original not yet publiſhed.

Deſigned by the AUTHOR for the Uſe of LEARNERS.

Hac via inſiſtendum eſt.

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MDCCXXXVII.

THE

PREFACE.

THE following Treatise containing the First Principles of Fluxions, though a posthumous Work, yet being the genuine Offspring (in an English *Dress) of the late Sir* Isaac Newton, *needs no other Recommendation to the Publick, than what that Great and Venerable Name will always carry along with it.*

Our Author in his Philosophy hath pushed his Researches through a prodigious Variety of remote Consequences and complicate Dependencies into the minutest Circumstances of Nature's Workings. This hath unavoidably rendered it very difficult and abstruse. Mathematicians of the first Character are obliged to study it with Care and Attention, and have Occasion for all their Skill in Algebra and Geometry to be able to comprehend the full Force and Extent of his Conclusions. Hence some previous Helps and Assistances have always been thought necessary to prepare young Students in the Mathematicks, before they attempt to enter upon these arduous Speculations, and several ingenious Gentlemen have usefully employed their Pens in drawing up Elements and Introductions to the Mathematical Principles of Philosophy, *as others also have applied themselves to supply the intermediate Steps and Conclusions (by him passed over in Silence) that lead to his sublime Speculations in Geometry.*

Although the Propositions in that Book for the sake of Elegance are demonstrated in the Synthetick Way according to the Manner of the Antients, whose Taste and Form of Demonstration our Author greatly Admired; yet it is well known that they were first discovered by the Use and Application of some Kind of Analysis. It cannot but be very acceptable therefore to all who have a Relish for these Enquiries to be furnished with that particular Method of Analyticks prepared by the Great Author himself, which He made use of in arriving at his sublime Discoveries.

It must be acknowledged that several Extracts and Specimens of this Method have been already published elsewhere, (particularly by Dr. Wallis *and Mr.* Jones;) *but as these were only incidentally delivered, or occasionally given out by the Author at the Importunity of his Friends, so they fall very much short of the Treatise here published: Wherein this noble Invention is digested into a just Method; the whole Extent and Compass of it, as far as he had improved it, is herein comprehended; all the Cases are taken in, and illustrated with a greater Variety of curious Instances, and the whole is enriched with a much larger Copia of choice Examples than is to be found any where else. In a Word, we have reason to believe that what is here delivered, is wrought up to that Perfection in which Sir* Isaac *himself had once intended to give it to the Publick *.*

The great Advantages of deriving our Knowledge from original Authors and Inventors, especially in these Subjects, are well understood by all who have made any Progress in them. One of which is, (and that no small one,) that we are hereby secure from that Puzzle and Perplexity into which Writers of an inferior Rank are perpetually plunging their Readers. But this is not what I mean. There are two distinguishing Excellencies of this Work, as it was intended for an

* Vid. Commer. Epist. pag. 149. *Lond.* Edit. 1722.

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Institution for Learners, which it may not be improper here more particularly to point out.

The first of these is an uncommon Condescension, and Familiarity with which the Great Author all along addresses himself to his Novitiate. He does not dictate as a Master, but rather as a Friend seems as it were to take him by the Hand, and in some Cases even to consult with him, as to the Fitness or Fairness of the Arts He makes use of to obtain his Conclusions. This Way of Teaching must needs have a good Effect. The young Student is irresistibly engaged to lend all his Attention to a Companion so agreeably instructive.

But the next Remark is what I principally intend, that whilst the Author is teaching that Art which He invented, He does in the mean time teach the Art of Invention itself. I mean He discovers those particular Endowments and Acquisitions by which he attained to so great an Eminence in that extraordinary Art.

The first of these I shall take Notice of is an accurate and comprehensive Knowledge of his Subject. The Subject here is Quantity, and what an immense Treasure of Learning he had laid up in his Mind, and throughly digested of all, even the most curious and latent Properties of Quantity, appears in every Page of this Work. One is fully convinced that He must have viewed it in all Lights, and considered it in all Relations; especially such as arise from the Conception of its being generated by local Motion. Hence proceeds that Variety of Solutions to answer all Difficulties that arise. Hence likewise he was able, in every particular Case, to supply that Property which was fittest for his Purpose, and which would resolve the Problem in the most simple and elegant Manner.

In the next place one cannot but observe the great natural Talent he had of discerning the several curious Analogies that obtain between the corresponding relations of Quantities of different Species. From this

Source

Source He derived those ingenious and useful Hints, that were afterwards improved by him, into the noblest Inventions and most sublime Theories. I am very sensible, that for his singular Sagacity this way He was much indebted to Nature. But I am apt to believe most Persons are endued with a good share of it; such I mean as have a Capacity and Genius for Mathematical Speculations. And I am persuaded, that our great Admiration of others, whom we see eminently conspicuous for this Talent, arises more from our own Neglect to cultivate the Bounty of Nature to us, than from any extraordinary Difference there is in her Gift.

This naturally leads me to the last Thing I shall observe on this Head. And that is our Author's unwearied Diligence and incessant Application in improving the Hints and Conceptions he had once formed, to the highest Perfection. He pusheth his Invention through all the Difficulties that can arise, extends it to all the Varieties of Cases that can happen, and at last applies it to the most curious Purposes.

These distinguishing Excellencies in this elementary Treatise of the greatest Master of Mathematical Learning that perhaps ever appeared in the World, I thought it not amiss to mention in this Place: As I conceive they afford the strongest Motives and most powerful Incitements to His Disciples, to follow their Great Leader in those Steps, by which He attained to the highest pitch of Human Glory.

I shall now proceed to take a short View of the Body of the Work, which may be divided into these two principal Members. The first is, the Method of Fluxions in its general Sense; and the other is the Application of this Method to the Geometry of Curve Lines. The first of these may again be subdivided into two Parts. The first of which contains the Doctrine of infinite Series, and is an Introduction to the other, wherein is delivered the Method of Fluxions peculiarly so called.

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In the first Part or the Method of Infinite Series, the Author very much enlarges the Boundaries of Analyticks by introducing into Algebra or specious Arithmetick a new Way of expressing Universal Radicals, (*such as* $\sqrt{rr-cc}$,) *by an infinite Series of simple Terms, which continually approach towards the true Value of the Roots, and if infinitely continued will be equal to them, and therefore may be used instead of them. He begins with pointing out the particular Analogy that would, if attended to, naturally furnish the Hint for this Improvement,* viz. *the Conformity there is between the Relation of Decimal Fractions to Vulgar Arithmetick, and that of Infinite Series to Common Algebra; and He explains the manner of this Correspondence. He has not given us here the particular Occasion which led him into the Road of improving the Hint. This was beside his present Purpose. But because, as I have already observed, a very good Use may be made of such Histories, and especially as this is communicated by the Author himself *, I shall therefore give it the Reader in* English *as follows.*

'*Not long after I had entered upon the Study of the Mathematicks, whilst I was perusing the Works of Our Celebrated Dr.* Wallis, *and considering the Series (of Universal Roots) by the Interpolation of which He exhibits the Area of the Circle and Hyperbola, for instance, in this Series of Curves whose Base or common Axis call* $=x$, *and the successive Ordinates call* $\overline{1-xx}|^{\frac{0}{2}} \cdot \overline{1-xx}|^{\frac{1}{2}} \cdot \overline{1-xx}|^{\frac{2}{2}} \cdot \overline{1-xx}|^{\frac{3}{2}} \cdot \overline{1-xx}|^{\frac{4}{2}} \cdot \overline{1-xx}|^{\frac{5}{2}}$, &c. *I observed that if the Areas of the Alternate Curves which are* x, $x-\frac{1}{3}x^3$, $x-\frac{2}{3}x^3+\frac{1}{5}x^5$, $x-\frac{3}{3}x^3+\frac{3}{5}x^5-\frac{1}{7}x^7$, &c. *could be interpolated, we should by this means obtain the Areas of the intermediate ones; the first of which* $\overline{1-xx}|^{\frac{1}{2}}$ *is the Area of the Circle.*

* Vid. Lib. supra cit. pag. 143.

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'*In order to this, first it was obvious that in each* '*of these Series the first Term was* x; *that the second* '*terms* $\frac{0}{3}x^3$, $\frac{1}{3}x^3$, $\frac{2}{3}x^3$, $\frac{3}{3}x^3$, &c. *were in an Arith-* '*metical Progression, and consequently the two first* '*Terms of the Series to be interpolated must be* $x-\frac{\frac{1}{2}x^3}{3}$, $x-\frac{\frac{3}{2}x^3}{3}$, $x-\frac{\frac{5}{2}x^3}{3}$, &c.

'*Now for the Interpolation of the rest, I considered* '*that the Denominators* 1, 3, 5, 7, &c. *were (in all* '*of them) in Arithmetical Progression, and consequently* '*the whole Difficulty consisted in finding out the numeral* '*Co-efficients. But these in the alternate Areas, which* '*are given, I observed were the same with the Figures* '*of which the several ascending Powers of the Number* '11 *consist*, viz. 11^0, 11^1, 11^2, 11^3, 11^4, &c. '*that is first* 1; *the second* 1,1; *the third* 1,2,1; *the* '*fourth* 1,3,3,1; *the fifth* 1,4,6,4,1, &c.

'*I applied myself therefore to seek for a Method by* '*which the two first Figures of these Series might be* '*derived from the rest; and I found, that if for the se-* '*cond Figure or numeral term we put m, the rest of* '*the terms will be produced by the continual Multipli-* '*cation of the Terms of this Series* $\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}$, &c.

'*For instance; Let the second Term m be put equal* '*to* 4, *and there will arise* $4\times\frac{m-1}{1}$, *that is* 6; *which* '*is the third Term. The fourth Term will be* $6\times\frac{m-2}{3}$, '*that is* 4. $4\times\frac{m-3}{4}=1$, *is the fifth Term; and the* '*sixth is* $4\times\frac{m-4}{1}=0$. *Which shews the Series is here* '*terminated in this Case.*

'*This being found I applied it as a Rule to in-* '*terpolate the abovementioned Series. And since in* '*the Series which will express the Circle, the second* *term*

'*term was found to be* $\frac{\frac{1}{2}x^3}{3}$. *Therefore I put m*
'$=\frac{1}{2}$, *and there was produced the Terms* $\frac{1}{2}\times\frac{\frac{1}{2}-1}{2}$ *or*
'$-\frac{1}{8}$; $-\frac{1}{8}\times\frac{\frac{1}{2}-2}{3}$ *or* $+\frac{1}{16}$; $+\frac{1}{16}\times\frac{\frac{1}{2}-3}{4}$ *or* $-$
'$\frac{5}{128}$, *and so on* in infinitum. *Hence I discovered*
'*that the Area sought of the Segment of the Circle is*
'$x-\frac{\frac{1}{2}x^3}{3}-\frac{\frac{1}{8}x^5}{5}-\frac{\frac{1}{16}x^7}{7}-\frac{\frac{5}{128}x^9}{9}$, *&c.*

'*In the same manner the Areas to be interpolated of*
'*the other Curves might be produced, as might also the*
'*Area of the Hyperbola and of the rest of the alternate*
'*Curves in this Series,* $\overline{1+xx}|^{\frac{0}{2}}$, $\overline{1+xx}|^{\frac{1}{2}}$, $\overline{1+xx}|^{\frac{2}{2}}$,
'$\overline{1+xx}|^{\frac{3}{2}}$, *&c.*

'*By the same Method likewise other Series might be*
'*interpolated, and that too if they should be taken at*
'*the distance of two or more intervals.*

'*This was the way by which I first opened an En-*
'*trance into these Speculations, which I should not have*
'*remembred, but that in turning over my Papers a few*
'*Weeks ago, I accidentally cast my Eyes upon those re-*
'*lating to this Matter.*

'*When I had proceeded thus far, it immediately*
'*occurred to me, that the Terms* $\overline{1-xx}|^{\frac{0}{2}}$, $\overline{1-xx}|^{\frac{2}{2}}$,
'$\overline{1-xx}|^{\frac{4}{2}}$, $\overline{1-xx}|^{\frac{6}{2}}$, *&c. that is* 1, $1-xx$, $1-2xx$
'$+x^4$, $1-3xx+3x^4-x^6$, *&c. might be interpolated*
'*in the same manner as I had done the Areas generated*
'*by them, and for this there needed nothing else, but only*
'*to leave out the Denominators* 1, 3, 5, 7, &c. *in the*
'*Terms that express the Areas; that is, the Co-efficients*
'*of the Terms of the Quantity to be interpolated* ($\overline{1-xx}|^{\frac{1}{2}}$,
'*or* $\overline{1-xx}|^{\frac{3}{2}}$; *or universally* $\overline{1-xx}|^{m}$,) *will be obtained by*
the

'*the continual multiplication of the terms of this Series*
'$m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, &c.

'*Thus (for Example)* $\overline{1-xx}|^{\frac{1}{2}} = 1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6$, '&c. *and* $\overline{1-xx}|^{\frac{3}{2}} = 1-\frac{3}{2}x^2+\frac{3}{8}x^4+\frac{1}{16}x^6$, &c. *and* '$\overline{1-xx}|^{\frac{1}{3}} = 1-\frac{1}{3}xx-\frac{1}{9}x^4-\frac{5}{81}x^6$, *&c.*

' *Thus I discovered a general Method of reducing* '*Radicals into Infinite Series by the Rule * which I sent* '*in my last Letter, before I observed that the same thing* '*might be obtained by the Extraction of Roots.*

But after I had found out that method, this other way '*could not remain long unknown; for in order to prove* '*the Truth of these Operations, I multiplied* $1-\frac{1}{2}x^2$ '$-\frac{1}{8}x^4-\frac{1}{16}x^6$, &c. *into itself, and the product is* '$1-xx$, *all the Terms after these* in infinitum *vanish-* '*ing; and so* $1-\frac{1}{3}xx-\frac{1}{9}x^4-\frac{5}{81}x^6$, &c. *twice drawn* '*into itself produced* $1-xx$. *As this was a certain* '*Demonstration of the Truth of these Conclusions, so I* '*was thereby naturally led to try the Converse of it,* 'viz. *whether these Series that now were known to be* '*the Roots of the Quantity* $1-xx$ *might not be extracted* '*thence by the Rule for Extraction of Roots in Arithme-* '*tick; and upon trial I found it succeed to my Desire.*

'*I shall here set down the form of the Process in* '*Quadraticks.*

$$
\begin{array}{l}
1-xx\,(1-\frac{1}{2}xx-\frac{1}{8}x^4-\frac{1}{16}x^6, \text{ \&c.} \\
1 \\
\hline
0-xx \\
-xx+\frac{1}{4}x^4 \\
\hline
\quad -\frac{1}{4}x^4 \\
\quad -\frac{1}{4}x^4+\frac{1}{8}x^6+\frac{1}{64}x^8 \\
\hline
\qquad -\frac{1}{8}x^6-\frac{1}{64}x^8, \text{ \&c.}
\end{array}
$$

'*This being found I laid aside the Method of Inter-* '*polation, and assumed these Operations as a more ge-*

* He means the famous *Binomial* Theorem, since well known.

'*nuine*

'*nuine Foundation to proceed upon. In the mean time*
'*I was not ignorant of the Way of Reduction by Divi-*
'*vision, which was so much easier.*

'*Proceeding upon this Foundation, the next thing I*
'*attempted, was the Resolution of affected Equations*;
'*which I also obtained*, &c.

We have in this Account the Origin of the several Improvements the Author made in the new Way of Notation by Infinite Series: the several Branches of which are here disposed in Order and methodically digested. He first shews how to resolve by Division Fractions with multinomial Denominators. Then He proceeds to extract the Roots of Pure Powers; and lastly exhibits the Method for extracting those likewise of affected Equations. And whereas the Methods delivered before by Vieta, Oughtred, *and others, for this Operation in Numbers, were very intricate and tedious, He here supplies one much more easy and free from that Load of superfluous Terms with which theirs were incumbred.*

The Foundation being thus laid, He passeth on to the Method of Fluxions. This is the Body and principal Part of the Work. It is the distinguishing Character of our Author, that from a few plain and obvious Principles He deduceth the most surprising Conclusions; and this Part of His Character no where appears to greater Advantage than in the Invention of His Method of Fluxions. The Ancients had considered the Area of a Rectangle as produced by the Motion of one of its Sides along the other. Our Author extends this Principle to all Kinds of mathematical Quantities. The Conception is very easy and natural: We see by continual Experience that all Kinds of Figures are actually described by the Motion of Bodies. But it is evident, that Quantities generated in this manner in a given Time become greater or less, in Proportion as the Velocity with which they are generated is greater or less. These were the Considerations that led the

Author to apply himſelf to the finding out of the Magnitudes of Finite Quantities by the Velocities of their generating Motions, which gave riſe to the Method of Fluxions.

The whole Method is here reduced to theſe two Problems : 1. The length of the Space deſcribed being continually given, to find the Velocity of the Motion at any time propoſed. *And* 2. *the Converſe of this.* The Velocity of the Motion being continually given, to find the Length of the Space deſcribed at any time propoſed.

In the Solution of the firſt Problem, as He is to find the comparative Velocities of Quantities, every thing is therefore ſuppoſed to be brought to an Equation. Then He ſhews how to reſolve it in its full extent, by multiplying the Terms by any arithmetical Progreſſion whatſoever ; hence an infinite Variety of Solutions may be obtained, ſo that we may always furniſh ſuch as beſt ſuit every particular Caſe. Then he ſhews how to find the Relation of the Fluxions, when the Equation involves ſurd Quantities, or even ſuch as are Geometrically irrational. Laſtly He demonſtrates all by the Method of Moments, which he here thus defines. Moments are the indefinitely ſmall Parts of flowing Quantities, by the Acceſſion of which in indefinitely ſmall Portions of Time, they are continually increaſed. Moments we ſee then are the indefinitely little Parts of finite Quantities ; that is, are leſſer than any Quantity that can be aſſigned. The ſame thing is meant, when it is ſaid they bear no Proportion to finite Quantities, or in Compariſon of them are nothing, and therefore may be rejected as ſuch. This Way of Demonſtration has been always received as juſt and legitimate, being founded upon this allowed Principle, that Quantity may be diminiſhed in infinitum, *or ſo far, as to become leſs than any finite or aſſignable Quantity whatſoever. All this is clear and intelligible concerning theſe Moments. And this is all that is neceſſary for any Uſe our Author makes of them ; and there-*

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fore (we may presume) this is the whole likewise of what He would be understood to mean by them. Doubtless many Difficulties will arise to such as busy themselves in making Enquiries into the precise Magnitude, the exact Form and Nature of infinite Quantities. In all our Reasonings about Infinity, there are certain Bounds set to our finite and limited Capacities, beyond which all is Darkness and Confusion. And it is the distinguishing Mark of true Philosophy to know where to stop. This is certain, we can know nothing of it but by Comparison only. However, such Conclusions as are fairly deduced from Principles taken in a Sense that we can comprehend, ought not to be rejected, on account of any Difficulties that may arise for want of a complete and adequate Understanding of the whole Extent and Nature of such Principles. In the mean time I cannot but observe, that our Author was greatly averse to Disputes upon any Account, and it was owing to his being unexpectedly drawn into one concerning his Opticks, that he laid aside the Design he had then of publishing this very Treatise of the Method of Fluxions. But to return.

The Author's next Problem is, an Equation being proposed including the Fluxions of Quantities, to find the Relation of these Quantities to one another. *And here because the Operation is easy and may sometimes be of use, he first gives the Solution in a particular Case. Then he proceeds to the general Solution wherein he comprehends the whole Compass of this most difficult Problem; shewing in all Cases how to obtain the Fluent either in finite Terms, or when that cannot be done, at least in an infinite Series. He has contrived many curious Processes for these Solutions, and often shews how the Fluent may be found an infinite Variety of Ways. But whereas in the fluential Equation thus obtained, there often comes out one or more Terms that are infinite, such as* $\frac{d}{0}$, *he has also provided an Expedient for this*

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Difficulty, which is the Transmutation of the flowing Quantity into another compounded of the said flowing Quantity, and a given one; by which means such infinite Quantity becomes finite, though consisting of Terms infinite in number.

In the latter Part, the Usefulness and Excellence of this Method is shewn by a successful Application of it to the making of several Improvements in the Geometry of Curve-Lines. But for these, that I may not repeat the same Things over again, I shall refer the Reader to the Contents. Observing only thus much in general; that as the Problem for determining the Quality of the Curvature of Curves is entirely new; so in such Speculations as have been already considered by others, the Reader will find all the Investigations and Constructions contrived with that beautiful Simplicity and Elegance which was peculiar to our Author. Lastly, I must not omit to take notice, that every thing is here performed without having recourse to second Fluxions: And He hath adjoined a Scholium to Prob. 9. *wherein is delivered a Theorem, by the help of which they may be managed as first Fluxions, and so their Fluents may be found by the Tables at that Problem.*

This is the Substance of the Work as we have it at present. It must be acknowledged that Sir Isaac *left it unfinished, and the first Occasion of His laying it aside I have already mentioned. The ingenious Dr.* Pemberton * *has acquainted us that he had once prevailed with Him to complete his Design and let it come abroad. But as Sir* Isaac's *Death unhappily put a stop to that Uudertaking, I shall esteem it none of the least Advantages of the present Publication, if it may prove a means of exciting that Honorable Gentleman, who is possessed of his Papers, to think of communicating them to some able Hand; that so the Piece may at last come out perfect and entire.*

* *Vid.* Preface to His View of Sir *Isaac Newton*'s Philosophy.

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PROBLEM

THE

OF THE

Method of FLUXIONS

AND

INFINITE SERIES.

Introduction: Or, the Resolution of Equations by Infinite Series.

HAVING obſerv'd that moſt of our modern Geometricians neglecting the ſynthetical Method of the Ancients, have applied themſelves chiefly to the analytical Art, and by the Help of it have overcome ſo many and ſo great Difficulties, that all the Speculations of Geometry ſeem to be exhauſted, except the Quadrature of Curves, and ſome other things of a like Nature which are not yet brought to Perfection: To this End I thought it not amiſs, for the ſake of young Students in this Science, to draw up the following Treatiſe; wherein I have endeavoured to enlarge the Boundaries of Analyticks, and to make ſome Improvements in the Doctrine of Curve Lines.

The great Conformity there is between the ſeveral Operations of the ſame Kind in Species and in common Numbers is obvious to every Body; indeed there ſeems to be no difference between them, except only in the Marks or Characters made uſe of in each. Upon this account I was very much ſurpris'd that a Method of transferring the lately invented Doctrine of Decimal Fractions in like manner to Species, had not been thought of, unlefs in the fingle inftance of the Quadrature of the Hyperbola by *Mercator*, and the rather fo, fince by this means a Way would have been opened to higher and more abftrufe Difcoveries, as will by and by appear.

This Doctrine of Species in an Infinite Series, bears the fame refpect to common Algebra, that the Method of decimal Fractions does to vulgar Arithmetick, and therefore the Operations of Addition, Subftraction, Multiplication, Divifion and Extraction of Roots here may be eafily learn'd from thence, if the Learner, whom we fuppofe pretty well fkill'd in decimal Arithmetick and the vulgar Algebra, duly obferves the Correfpondence that obtains between decimal Fractions and algebraick Terms infinitely continued; for as in Numbers the Places towards the right Hand continually decreafe in a decuple or fubdecuple Proportion, fo it is refpectively in Species when the Terms are difpos'd (as is often directed in what follows) in an uniform Progreffion infinitely continued according to the Order of the Dimenfions of any Numerator or Denominator: And as the advantage of Decimals is this, that all vulgar Fractions and Radicals being reduced to them, in fome meafure acquire the Nature of Integers, and may be manag'd as fuch, fo it is a Convenience attending infinite Series in Species that all Kinds of complicate Terms (fuch as Fractions whofe Denominators are compounded, the Roots of compound Quantities or of affected Equations, and the like) may be reduced to the Clafs of fimple Quantities, *i. e.* to an infinite Series of Fractions whofe Numerators and Denominators are fimple Terms, which will thus be freed from thofe Difficulties that in their original Form feem'd almoft infuperable. In the firft place therefore I fhall fhew how thefe Reductions are to be perform'd, or how any compound

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Quantities may be reduc'd to ſuch ſimple Terms, eſpecially when the Methods of computing are not obvious: After which I ſhall apply this Analyſis to the Solution of Problems.

Reduction by Diviſion and Extraction of Roots will be plain from the following Examples, if you compare the like Methods of Operation in Decimals and in ſpecious Arithmetick.

Examples by Diviſion.

The Fraction $\frac{aa}{b+x}$ being propoſed, divide aa by $b+x$ in the following Manner.

$$b+x)\ aa+0\ \left(\frac{aa}{b}-\frac{aax}{b^2}+\frac{aax^2}{b^3}-\frac{aax^3}{b^4}+\frac{aax^4}{b^5}\ \&c.\right.$$

$$aa+\frac{aax}{b}$$

$$0-\frac{aax}{b}+0$$

$$-\frac{aax}{b}-\frac{aax^2}{b^2}$$

$$0+\frac{a^2x^2}{b^2}+0$$

$$+\frac{a^2x^2}{b^2}+\frac{a^2x^3}{b^3}$$

$$0-\frac{a^2x^3}{b^3}+0$$

$$-\frac{a^2x^3}{b^3}-\frac{a^2x^4}{b^4}$$

$$0+\frac{a^2x^4}{b^4}\ \&c.$$

The Quotient therefore is $\frac{aa}{b}-\frac{a^2x}{b^2}+\frac{a^2x^2}{b^3}-\frac{aax^3}{b^4}+\frac{aax^4}{b^5}$ &c. which Series infinitely continued is

 equivalent

equivalent to $\frac{aa}{b+x}$ Or making x the first Term of the Divisor in this manner $x+b)\ aa+0\ (\frac{aa}{x}-\frac{aab}{x^2}+\frac{a^2b^2}{x^3}-\frac{a^2b^3}{x^4}$ &c. the Quotient will be found as in the foregoing Process.

In like manner the Fraction $\frac{1}{1+xx}$ will be reduced to $1-x^2+x^4-x^6+x^8$ &c. or to $x^{-2}-x^{-4}+x^{-6}-x^{-8}$ &c. and the Fraction $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}$ will be reduced to $2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^2+34x^{\frac{5}{2}}$ &c.

Here it must be observed that I make use of $x^{-1},\ x^{-2},\ x^{-3},\ x^{-4}$ &c. for $\frac{1}{x},\frac{1}{x^2},\frac{1}{x^3},\frac{1}{x^4}$, &c. as also of $x^{\frac{1}{2}},\ x^{\frac{3}{2}},\ x^{\frac{5}{2}},\ x^{\frac{1}{3}},\ x^{\frac{2}{3}}$ &c. for $\sqrt{x}$, $\sqrt{x^3}$, $\sqrt{x^5}$, $\sqrt[3]{x}$, $\sqrt[3]{x^2}$ &c. and of $x^{-\frac{1}{2}},\ x^{-\frac{2}{3}},\ x^{-\frac{1}{4}}$ &c. for $\frac{1}{\sqrt{x}},\frac{1}{\sqrt[3]{x^2}},\frac{1}{\sqrt[4]{x}}$, which way of Notation is drawn from the Rule of Analogy, as may be apprehended from these and such like geometrical Progessions, *viz.* $x^3,\ x^{\frac{5}{2}},\ x^2,\ x^{\frac{3}{2}},\ x,\ x^{\frac{1}{2}},\ x^0$ (or 1) $x^{-\frac{1}{2}},\ x^{-1},\ x^{-\frac{3}{2}},\ x^{-2}$ &c.

In the same manner for $\frac{aa}{x}-\frac{aab}{x^2}+\frac{aab^2}{x^3}$ &c. may be wrote $aax^{-1}-a^2bx^{-2}+a^2b^2x^{-3}$ &c. Instead of $\sqrt{aa-xx}$ may be wrote $\overline{aa-xx}|^{\frac{1}{2}}$; and $\overline{aa-xx}|^2$ instead of the Square of $aa-xx$; and $\overline{\frac{ab^2-y^3}{by+yy}}|^{\frac{1}{3}}$ instead of $\sqrt[3]{\frac{ab^2-y^3}{by+yy}}$; and the like of others. Hence we may not improperly distinguish Powers into affirmative and negative, integral and fractional.

Examples of Reduction by Extraction of Roots.

The Quantity $aa+xx$ being proposed, you may thus extract its square Root.

$$
\begin{array}{l}
aa+xx\ \left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\frac{7x^{10}}{256a^9}-\frac{21x^{12}}{1024a^{11}}\right. \\
aa \\
\hline
0+xx \\
\quad +xx+\frac{x^4}{4a^2} \\
\hline
\qquad -\frac{x^4}{4a^2} \\
\qquad -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\
\hline
\qquad\qquad +\frac{x^6}{8a^4}-\frac{x^8}{64a^6} \\
\qquad\qquad +\frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}+\frac{x^{12}}{256a^{10}} \\
\hline
\qquad\qquad\qquad -\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}-\frac{x^{12}}{256a^{10}} \\
\qquad\qquad\qquad -\frac{5x^8}{64a^6}-\frac{5x^{10}}{128a^8}+\frac{5x^{12}}{512a^{10}}\ \&c. \\
\hline
\qquad\qquad\qquad\qquad +\frac{7x^{10}}{128a^8}-\frac{7x^{12}}{512a^{10}}\ \&c. \\
\qquad\qquad\qquad\qquad +\frac{7x^{10}}{128a^8}+\frac{7x^{12}}{256a^{10}} \\
\hline
\qquad\qquad\qquad\qquad\qquad -\frac{21x^{12}}{512a^{10}}\ \&c.
\end{array}
$$

So that the Root is found to be $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}$ &c. Where it may be observed, that towards the End of the Operation I neglect all those Terms whose Dimensions would exceed the Dimensions of the last Term, to which only I intend to continue the Root suppose to $\frac{x^{12}}{a^{11}}$

Here

Here alſo the Order of the Terms may inverted in this manner $xx+aa$ in which Caſe the Root will be $x+\frac{aa}{2x}-\frac{a^4}{8x^3}+\frac{a^6}{16a^5}-\frac{5a^8}{128a^7}$ &c. likewiſe the Root of $x-xx$ is $x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{3}{2}}-\frac{1}{8}x^{\frac{5}{2}}-\frac{1}{16}x^{\frac{7}{2}}$ &c. The Root of $aa+bx-xx$ is $a+\frac{bx}{2a}-\frac{xx}{2a}-\frac{b^2x^2}{8a^3}$ &c. And laſtly the Root of $\frac{1+axx}{1-xbx}$ is $\frac{1+\frac{1}{2}ax^2-\frac{1}{8}a^2x^4+\frac{1}{16}a^3x^6}{1-\frac{1}{2}bx^3-\frac{1}{8}b^2x^4-\frac{1}{16}b^3x^6}$ &c. which by dividing becomes $1+\overline{\frac{1}{2}b+\frac{1}{2}a}\times x^2+\overline{\frac{3}{8}b^2+\frac{1}{4}ab-\frac{1}{8}a^2}\times x^4+\overline{\frac{5}{16}b^3+\frac{3}{16}ab^2-\frac{1}{16}a^2b+\frac{1}{16}a^3}\times x^6$, &c.

But theſe Operations may very often be abbreviated by a due Preparation, as in the foregoing Example to find $\sqrt{\frac{1+axx}{1+bxx}}$, if the Numerator and Denominator had not been the ſame, I might have multiplied each by $\sqrt{1-bxx}$ which would have produced $\sqrt{\frac{1+ax^2-abx^4 \atop -b}{1-bxx}}$ and the reſt of the Work might have been perform'd by extracting the Root of the Numerator only, and then dividing by the Denominator.

Hence I conceive it will ſufficiently appear by what means any other Roots may be extracted, and how any compound Quantities, though never ſo much entangled with Radicals or compound Denominators (ſuch for inſtance as this $x^3+\frac{\sqrt{x}-\sqrt{1-xx}}{\sqrt{axx+x^3}}-\frac{\sqrt[5]{x^3}+2x^5-x^{\frac{3}{2}}}{\sqrt[3]{x^3+xx}-\sqrt{2x-x^{\frac{2}{3}}}}$) may be reduced to infinite Series conſiſting of ſimple Terms.

Of the Reduction of affected Equations.

As to affected Equations we muſt be ſomething more particular in explaining how their Roots are

to

to be reduced to ſuch Series as theſe, becauſe their Doctrine in Numbers as hitherto deliver'd by Mathematicians is very perplex'd, and incumber'd with ſuperfluous Operations, ſo as not to afford proper Specimens for performing the work in Species. I ſhall therefore firſt ſhew how the Reſolution of affected Equations may be compendiouſly performed in Numbers, then I ſhall apply the ſame to Species.

Let this Equation $y^3-2y-5=0$ be propoſed to be reſolv'd, and let 2 be a Number (any how found) which differs from the true Root leſs than by a tenth part of itſelf, then I make $2+p=y$, and ſubſtitute $2+p$ for y in the given Equation, by which is produced a new Equation $p^3+6p^2+10p-1=0$ whoſe Root is to be ſought for that it may be added to the Quote. Thus rejecting p^3+6p^2 becauſe of its Smallneſs, the remaining Equation $10p-1=0$ or $p=0,1$ will approach very near to the Truth. Therefore I write this in the Quote, and ſuppoſe $0,1+q=p$, and ſubſtitute this fictitious Value of p as before, which produces $q^3+6,3q^2+11,23q+0,361=0$, and ſince $11,23q+0,061=0$ is near the truth, or $q=0,0054$ nearly (*i. e.* dividing 0,061 by 11,23 till ſo many Figures ariſe as there are places between the firſt ſignificant Figure of this and of the principal Quote excluſively, as here there are two ſuch Places between 2 and 0,005) I write 0,0054 in the lower part of the Quote as being negative, and ſuppoſing $-0,0054+r=q$, I ſubſtitute this as before, and thus I continue the Operation as far as I pleaſe after the manner exhibited in the following Table.

$y^3 - 2y - 5 = 0$		$+2,10000000$
		$-0,00544852$
		$+2,09455148$, &c. $= y$
$2 + p = y$	y^3	$+8 + 12p + 6p^2 + p^3$
	$-2y$	$-4 - 2p$
	-5	-5
	The Sum	$-1 + 10p + 6p^2 + p^3$
$0,1 + q = p$	$+p^3$	$+0,001 + 0,03q + 0,3q^2 + q^3$
	$+6p^2$	$+0,06 + 1,2 + 6$
	$+10p$	$+1 + 10$
	-1	$+1$
	The Sum	$+0,061 + 11,23q + 6,3q^2 + q^3$
$-0,0054 + r = q$	q^3	$-0,000000157464 + 0,00087480r - 0,0162r^2 + r^3$
	$+6,3q^2$	$+0,000183708 - 0,06804 + 6,3$
	$+11,23q$	$-0,060642 + 11,23$
	$+0,061$	$+0,061$
	The Sum	$+0,0005416 + 11,162r + 6,3r^2$
$-0,00004852 + s = r$		

But

But the Work may be much ſhortned towards the End, eſpecially in Equations of many Dimenſions by this Method. Having firſt determin'd how far you intend to extract the Root, count ſo many Places after the firſt Figure of the Co-efficient of the laſt Term but one, of the Equations that reſult on the right Side of the Table, as there remain places to be fill'd up in the Quote, and reject the Decimals that follow. But in the laſt Term the Decimals may be neglected after ſo many more Places as there are decimal Places fill'd up in the Quote, and in the antepenultimate Term reject all that are after ſo many fewer places; and ſo on by proceeding arithmetically according to that Interval of Places: Or, which is the ſame thing, you may cut off every where ſo many Figures as in the penultimate Term, ſo that their loweſt Places may be in arithmetical Progreſſion according to the Series of the Terms, or muſt be conceiv'd to be fill'd up with Cyphers when it happens otherwiſe. Thus in the preſent Example, if I deſired to continue the Quote no farther than to the eighth Place of Decimals, when I ſubſtituted 0,0054 $+r$ for q, where four decimals are completed in the Quote, and as many more remain to be found, I might have omitted the Figures in the five inferior Places; which therefore I have mark'd or cancell'd by little lines drawn through them; and indeed I might have omitted the firſt term r^3, although its Co-efficient be 0,99999; thoſe Figures therefore being expung'd, for the following operation there ariſes the Sum $0{,}0005416+11{,}162r$, which by Diviſion continued as far as the Term preſcrib'd, gives 0,00004852 for r, which completes the Quote to the Period requir'd; then ſubtracting the negative Part of the Quote from the affirmative Part there ariſes 2,09455148 for the Root of the propos'd Equation.

It may likewise be observ'd that at the beginning of the Work, if I had doubted whether 0,1 $+p$ were a sufficient Approximation to the Root, instead of $10p-1=0$, I might have supposed $6p^2+10p-1=0$, and so have wrote the first Figure of its Root in the Quote as being near to nothing; and in this manner it may be convenient to find the second or even the third Figure of the Root, when in the secondary Equation, about which you are conversant, the Square of the Co-efficient of the penultimate Term is not ten times greater than the Product of the last Term multiplied into the Co-efficient of the antepenultimate Term: And indeed you will often save some Pains, especially in Equations of many Dimensions, if you seek for all the Figures to be added to the Quote in this manner, that is, if you extract the lesser Root out of the three last Terms of its secondary Equation: For thus you will obtain at every time as many Figures again in the Quote.

And now from the Resolution of numeral Equations I proceed to explain the like Operations in Species; concerning which it will be necessary to premise the following Observations.

First, That some one of the Species or literal Co-efficients, if there are more than one, should be distinguished from the rest, which either is or may be supposed to be much the least or greatest of all, or nearest to a given Quantity: The Reason of which is that because of its Dimensions continually encreasing in the Numerators or the Denominators of the Terms of the Quote, those Terms may grow less and less, and therefore the Quote may constantly approach to the Root required; as may appear from what is said before of the Species x in the Examples of Reduction by Division and Extraction of Roots, and hereafter for this Species I shall generally make use of x or z, as also

y, p,

y, p, q, r, s, *&c.* for the radical Species to be extracted.

Secondly, When any complex Fractions or surd Quantities happen to occur in the proposed Equation, or to arise afterwards in the Process, they ought to be removed by such Methods as are sufficiently known to Analysts. As if we should have $y^3+\frac{bb}{b-x}y^2-x^3=0$, multiply by $b-x$ and from the Product $by^3-xy^3+b^2y^2-bx^3+x^4=0$, extract the Root y. Or we might suppose $y\times\overline{b-x}=v$, and then writing $\frac{v}{b-x}$ for y, we should have $v^3+b^2v^2-b^3x^3+3b^2x^4-3bx^5+x^6=0$, whence extracting the Root v, we might divide the Quote by $b-x$ in order to obtain y: Also if the Equation $y^3-xy^{\frac{1}{2}}+x^{\frac{4}{3}}=0$ were proposed, we might put $y^{\frac{1}{2}}=v$, and $x^{\frac{1}{3}}=z$, and so writing vv for y and z^3 for x, there will arise $v^6-z^3v+z^4=0$; which Equation being resolved, y and x may be restored, for the Root will be found $v=z+z^3+6z^5$, *&c.* and restoring y and x, we have $y^{\frac{1}{2}}=x^{\frac{1}{3}}+x+6x^{\frac{5}{3}}$, *&c.* then squaring $y=x^{\frac{2}{3}}+2x^{\frac{4}{3}}+13x^2$, *&c.*

After the same manner if there should be found negative Dimensions of x and y, they may be remov'd by multiplying by the same x and y, as if we had the Equation $x^3+3x^2y^{-1}-2x^{-1}-16y^{-3}=0$, multiply by x and y^3, and there will arise $x^4y^3+3x^3y^2-2y^3-16x=0$, and if the Equation were $x=\frac{aa}{y}-\frac{2a^3}{y^2}+\frac{3a^4}{y^3}$, by multiplying both parts into y^3 there would arise $xy^3=a^2y^2-2a^3y+3a^4$. And so of others.

Thirdly, When the Equation is thus prepared, the Work begins by finding the first Term of the Quote; concerning which, as also for finding the following Terms, we have this general Rule, when the

the indefinite Species x or z is suppofed to be fmall, to which Cafe the other two are reducible; *i. e. either when the faid Species is very great or when it nearly approaches to a given Quantity.*

Of all the Terms in which the radical Species (y, p, q or r, *&c.*) is not found, choofe the loweft in refpect of the dimenfions of the indefinite Species, (x, z, *&c.*) then choofe any other Term in which that radical Species is found, fuch as that the Progreffion of the Dimenfions of each of the foremen-tioned Species being continued from the Term firft affumed to this Term, may defcend as much as may be, or afcend as little as may be; and if there are any other Terms whofe Dimenfions may fall in with this Progreffion continued at pleafure, they muft be taken in likewife; laftly, from thefe Terms thus felected, and made equal to nothing, find the Value of the faid radical Species, and write it in the Quote.

But that this Rule may be more clearly apprehended, I will explain it farther by help of the annexed Diagram. Make the Right Angle BAC, and divide its Sides AB, AC into equal Parts; then by Perpendiculars rais'd from every Point in the Divifion, diftribute the angular Space into equal Squares or Parallelograms, which you may conceive to be denominated from the Dimenfions of the Species x and y, as they are here infcrib'd:

B

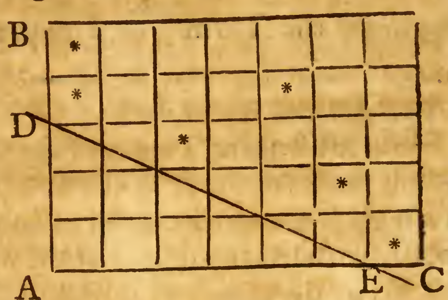
x^4	x^4y	x^4yy	x^4y^3	x^4y^4	x^4y^5	x^4y^6
x^3	x^3y	x^3yy	x^3y^3	x^3y^4	x^3y^5	x^3y^6
x^2	x^2y	x^2yy	x^2y^3	x^2y^4	x^2y^5	x^2y^6
x	$x\,y$	$x\,yy$	$x\,y^3$	$x\,y^4$	$x\,y^5$	$x\,y^6$
1	y	yy	y^3	y^4	y^5	y^6

A C

Then when any Equation is propos'd, mark fuch of the Parallelograms as correfpond to all the Terms, and let a Ruler be apply'd to two or perhaps more of the Parallelograms thus mark'd, of which

which let one be the loweſt in the left Hand Column at AB, and the other touching the Ruler towards the right Hand; and let all the reſt not touching the Ruler lie above it: Then ſelect thoſe Terms of the Equation which are repreſented by the Parallelograms that touch the Ruler, and from them find the Quantity to be put in the Quote.

Thus to extract the Root y out of the Equation $y^6 - 5xy^5 + \frac{x^3}{a} y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$, I mark the Parallelograms belonging to the Terms of this Equation with the Mark * as you ſee here done.



Then I apply the Ruler DE to the lower of the Parallelograms mark'd in the left Hand Column, and turning it round upwards towards the right Hand till it begins in like manner to touch another or perhaps more of the Parallelograms that are mark'd, I ſee that the Places ſo touch'd belong to x^3, x^2y^2, and y^6. Therefore from the Terms $y^6 - 7a^2x^2y^2 + 6a^3x^3$, as if equal to nothing (and moreover if you pleaſe reduced to $v^6 - 7v^2 + 6 = 0$ by making $y = v\sqrt{ax}$,) I ſeek the Value of y and find it to be fourfold $+\sqrt{ax}$, $-\sqrt{ax}$, $+\sqrt{2ax}$, and $-\sqrt{2ax}$; of which I may take any one for the initial Term of the Quote, according as I deſign to extract this or that Root of the given Equation. Thus having the Equation $y^5 - by^2 + 9bx^2 - x^3 = 0$, I chuſe the Terms $-by^2 + 9bx^2$, and thence I obtain $+3x$ for the initial Term of the Quote. And having $y^3 + axy + aay^3 - x^3 - 2a^3 = 0$, I make choice of $y^3 + a^2y - 2a^3$, and its Root $+a$ I write in the Quote. Alſo having $x^2y^5 - 3c^4xy^2 - c^5x^2 + c^7 = 0$

$=0$ I select $x^2y^5+c^7$, which gives $-\sqrt[5]{\frac{c^7}{x^2}}$ for the first Term of the Quote. And the like of others.

But when this Term is found, if its Power should happen to be negative, I depress the Equation by the same Power of the indefinite Species, that there may be no need of depressing it in the Resolution; and besides that the Rule hereafter delivered for the Suppression of superfluous Terms may be commodiously apply'd. Thus the Equation $8z^6y^3+az^6y^2-27a^9=0$, whose Root is to begin by the Term $\frac{3a^3}{2z^2}$, I depress by z^2, that it may become $8z^4y^3+az^4y^2-27a^9z^{-2}=0$, before I attempt the Resolution.

The subsequent Terms of the Quotes are deriv'd by the same Method, in the Progress of the Work, from their several secondary Equations, but commonly with less trouble. For the whole Affair is perform'd by dividing the lowest of the Terms affected with the indefinitely small species (x, x^2, x^3, *&c.*) without the radical Species (p, q, r, *&c.*), by the Quantity with which that radical Species of one Dimension only is affected without the other indefinite Species, and by writing the result in the Quote. So in the following Example the Terms $\frac{x}{4}$, $\frac{xx}{64a}$, $\frac{131x^3}{512a^2}$, *&c.* are produced by dividing a^2x, $\frac{1}{16}ax^2$, $\frac{131}{128}x^3$, *&c.* by $4aa$.

These things being premised, it remains now to exhibit the Praxis of Resolution. Let then the Equation $y^3+a^2y+axy-2a^3-x^3=0$ be propos'd to be resolv'd, and from its Terms $y^3+a^2y-2a^3=0$ being a fictitious Equation, by the third of the foregoing Premises I obtain $y-a=0$, and therefore I write $+a$ in the Quote, then because $+a$ is not the complete Value of y, I put $a+p=y$, and instead of y in the Terms of the Equation written in the Margin, I substitute fubftitute $a+p$, and the Terms refulting $(p^3 + 3ap^2 + axp$, *&c.*) I again write in the Margin, from which again, according to the third of the Premifes, I felect the Terms $+4a^2p+a^2x=0$ for a fictitious Equation, which giving $p=-\frac{1}{4}x$, I write $-\frac{1}{4}x$ in the Quote. Then becaufe $-\frac{1}{4}x$ is not the accurate Value of p, I put $-\frac{1}{4}x+q=p$, and in the marginal Terms for p I fubftitute $-\frac{1}{4}x+q$, and the refulting Terms $(q^3-\frac{3}{4}xq^2+3aq^2$, *&c.*) I again write in the Margin, out of which, according to the foregoing Rule, I again felect the Terms $4a^2q-\frac{1}{16}ax^2=0$ for a fictitious Equation, which giving $q=\frac{xx}{64a}$, I write $\frac{xx}{64a}$ in the Quote. Again fince $\frac{xx}{64a}$ is not the accurate Value of q, I make $\frac{xx}{64a}+r=q$, and fo inftead of q I fubftitute $\frac{xx}{64a}+r$ in the marginal Terms. And thus I continue the Procefs at pleafure, as the following Table exhibits to view.

$$y^3+a^2y-2a^3+axy-x^3=0.$$

$$y=a-\frac{x}{4}+\frac{xx}{64a}+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}, \&c.$$

$+a+p=y$	$+y^3$	$+a^3+3a^2p+3ap^2+p^3$
	$+axy$	$+a^2x+axp$
	$+a^2y$	$+a^3+a^2p$
	$-x^3$	$-x^3$
	$-2a^3$	$-2a^3$
$-\frac{1}{4}x+q=p$	$+p^3$	$-\frac{1}{64}x^3+\frac{3}{16}x^2q-\frac{3}{4}xq^2+q^3$
	$+3ap^2$	$+\frac{3}{16}ax^2-\frac{3}{2}axq+3aq^2$
	$+axp$	$-\frac{1}{4}ax^2+axq$
	$+4a^2p$	$-a^2x+4a^2q$
	$+a^2x$	$+a^2x$
	$-x^3$	$-x^3$
$+\frac{x^2}{64a}+r=q$	$+q^3$	*
	$-\frac{3}{4}xq^2$	*
	$+3aq^2$	$+\frac{3x^4}{4096a}$ * $+\frac{3}{32}x^2r+3ar^2$
	$+\frac{3}{16}x^2q$	$+\frac{3x^4}{1024a}$ * $+\frac{3}{16}x^2r$
	$-\frac{1}{2}axq$	$-\frac{1}{128}x^3-\frac{1}{2}axr$
	$+4a^2q$	$+\frac{1}{16}ax^2+4a^2r$
	$-\frac{65}{64}x^3$	$-\frac{65}{64}x^3$
	$-\frac{1}{16}ax^2$	$-\frac{1}{16}ax^2$

$$+4a^2-\tfrac{1}{2}ax+\tfrac{9}{32}x^2)+\tfrac{131}{128}x^3-\frac{15x^4}{4096a}(+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$$

If it were required to continue the Quote only to a certain Period, that x, for inſtance, ſhould not aſcend beyond a given Dimenſion; in ſubſtituting the Terms, I omit ſuch as I foreſee will be of no Uſe: For which this is the Rule, that after the firſt Term in the collateral Margin reſulting from every Quantity, no more are to be added on the right Hand, than there are Degrees of Dimenſion in the higheſt Term required in the Quote, above the Degrees of that firſt reſulting Term.

As

As in the present Example, if I desir'd that the Quote, (or the Species x in the Quote,) should ascend no higher than to four Dimensions, I omit all the Terms after x^4, and put only one after x^3: Therefore the Terms after the Mark * may be conceived to be expung'd. And thus the Work being continued till at last we come to the Terms $\frac{15x^4}{4096a} - \frac{131}{128}x^3 + 4a^2r - \frac{1}{2}axr$, in which p, q, r, or s, *&c.* representing the Supplement of the Root to be extracted, are only of one Dimension; we may find as many Terms ($+\frac{131x^3}{512a^2} + \frac{509x^4}{16384a^4}$) by Division, as we shall see wanting to compleat the Quote. So that at last we shall have $y = a - \frac{1}{4}x + \frac{xx}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$, *&c.*

For the sake of farther Illustration I shall propose another Example to be resolv'd. From the Equation $\frac{1}{5}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y - z = 0$, let the Quotient be found only to the fifth Dimension, and the superfluous Terms be rejected after the Mark, *&c.*

$\frac{1}{5}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y - z = 0.$		
$y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ &c.		
$z + p = y$	$+ \frac{1}{5}y^5$	$+\frac{1}{5}z^5$, &c.
	$- \frac{1}{4}y^4$	$-\frac{1}{4}z^4 - z^3p$, &c.
	$+ \frac{1}{3}y^3$	$+\frac{1}{3}z^3 + z^2p + zp^2$, &c.
	$- \frac{1}{2}y^2$	$-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$
	$+ y$	$+z \quad +p$
	$- z$	$-z$
$\frac{1}{2}z^2 + q = p$	$+ zp^2$	$+\frac{1}{4}z^5$, &c.
	$- \frac{1}{2}p^2$	$-\frac{1}{8}z^4 - \frac{1}{2}z^2q$, &c.
	$- z^3p$	$-\frac{1}{2}z^5$, &c.
	$+ z^2p$	$+\frac{1}{2}z^4 + z^2q$
	$- zp$	$-\frac{1}{2}z^3 - zq$
	$+ p$	$+\frac{1}{2}z^2 + q$
	$+ \frac{1}{5}z^5$	$+\frac{1}{5}z^5$
	$- \frac{1}{4}z^4$	$-\frac{1}{4}z^4$
	$+ \frac{1}{3}z^3$	$+\frac{1}{3}z^3$
	$- \frac{1}{2}z^2$	$-\frac{1}{2}z^2$
$1 - z + \frac{1}{2}z^2) \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 (\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$		

And thus if we propose the Equation $\frac{63}{2816}y^{11} + \frac{35}{1152}y^9 + \frac{5}{112}y^7 + \frac{3}{40}y^5 + \frac{1}{6}y^3 + y - z = 0$ to be resolv'd only to the ninth Dimension of the Quote, before the work begins we may reject the Term $\frac{63}{2816}y^{11}$; then as we operate we may reject all the Terms beyond z^9, beyond z^7 we may admit but one, and two only after z^5; because we may observe that the Quote ought always to ascend by the interval of two Units in this manner z, z^3, z^5, &c. Then at last we shall have $y = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$, &c.

And hence an Artifice is discover'd, by which Equations, though affected *in infinitum*, and consisting of an infinite Number of Terms, may however be resolved. And that is, before the Work begins all the Terms are to be rejected, in which the Dimension

Dimenſion of the indefinitely ſmall Species not affected by the radical Species exceeds the greateſt Dimenſion required in the Quote; or from which by ſubſtituting, inſtead of the radical Species, the firſt Term of the Quote found by the Parallelogram as before, none but ſuch exceeding Terms can ariſe. Thus in the laſt Example, I ſhould have omitted all the Terms beyond y^9, tho' they went on *ad infinitum*. And ſo in this Equation

$$0=\begin{cases} -8+z^2-4z^4+9z^6-16z^8, \ \&c. \\ +y \text{ in } z^2-2z^4+3z^6-4z^8, \ \&c. \\ -y^2 \text{ in } z^2-z^4+z^6-z^8, \ \&c. \\ +y^3 \text{ in } z^2-\frac{1}{2}z^4+\frac{1}{3}z^6-\frac{1}{4}z^8, \ \&c. \end{cases}$$

that the Cubic Root may be extracted only to four Dimenſions of z, I omit all the Terms in *infinitum* beyond $+y^3$ in $z^2-\frac{1}{2}z^4+\frac{1}{8}z^6$, and all beyond $-y^2$ in $z^2-z^4+z^6$, and all beyond $+y$ in z^2-2z^4, and beyond $-8+z^2-4z^4$. And therefore I aſſume this Equation only to be reſolved $\frac{1}{3}z^6y^3-\frac{1}{2}z^4y^3+z^2y^3-z^6y^2+z^4y^2-z^2y^2-2z^4y+z^2y-4z^4+z^2-8=0$ becauſe $2z^{-\frac{2}{3}}$ (the firſt Term of the Quote) being ſubſtituted inſtead of y in the reſt of the Equation depreſſed by $z^{\frac{2}{3}}$ gives every where more than four Dimenſions.

What I have ſaid of higher Equations may alſo be applied to Quadraticks. As if I deſir'd the Root of this Equation

$$0=\begin{cases} y^2 \\ -y \text{ in } a+x+\frac{x^2}{a}+\frac{x^3}{a^2}-\frac{x^4}{a^3}, \ \&c. \\ +\frac{x^4}{4a^2} \end{cases}$$

as far as the Period x^6. I omit all the Terms in *infinitum* beyond $-y$ in $a+x+\frac{x^2}{a}$, and aſſume only this Equation $y^2-ay-xy-\frac{x^2}{a}y+\frac{x^4}{4a^2}=0$. This I reſolve either in the uſual manner by making $y=\frac{1}{2}a$

 $+\frac{1}{2}x$

$+\frac{1}{2}x+\frac{x^2}{2a}-\sqrt{\frac{1}{4}a^2+\frac{1}{2}ax+\frac{3}{4}x^2+\frac{x^3}{2a}}$; or more expressly by the Method of affected Equations delivered before, by which we shall have $y=\frac{x^4}{4a^3}-\frac{x^5}{4a^4}*$, where the last required Term vanishes or becomes equal to nothing.

Now after that Roots are extracted to a convenient Period, they may sometimes be continued at pleasure only by observing the Analogy of the Series. So you may for ever continue this $z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4+\frac{1}{120}z^5$, (which is the Root of the infinite Equation $z=y+\frac{1}{2}y^2+\frac{1}{3}y^3+\frac{1}{4}y^4$, *&c.*) by dividing the last Term by these Numbers in order, 2, 3, 4, 5, 6, *&c.* And this $z-\frac{1}{6}z^3+\frac{1}{120}z^5-\frac{1}{5040}z^7+\frac{1}{362880}z^9$, *&c.* may be continued by dividing by these Numbers 2×3, 4×5, 6×7, 8×9, *&c.* Again the Series $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$, *&c.* may be continued at pleasure, by multiplying, the Terms respectively by these Fractions, $\frac{1}{2}$, $-\frac{1}{4}$, $-\frac{3}{6}$, $-\frac{5}{8}$, $-\frac{7}{10}$, *&c.* And so of others.

But in discovering the first Term of the Quotient, or sometimes the second or third, there may still remain a difficulty to be overcome; for its Value sought for as before, may happen to be surd, or the inextricable Root of an high affected Equation. Which when it happens, provided it be not also impossible, you may represent it by some Letter, and then proceed as if it were known, as in the Example $y^3+axy+a^2y-x^3-2a^3=0$, if the Root of this Equation $y^3+a^2y-2a^3$ had been surd or unknown, I should have put any Letter b for it, and then have perform'd the Resolution as follows; suppose the Quote found only to the third Dimension.

$+y^3$

$y^3+a^2y+axy-2a^3-x^3=0$, make $a^2+3b^2=c^2$, then

$y=b-\frac{abx}{c^2}+\frac{a^4bx^2}{c^6}+\frac{x^3}{c^2}+\frac{a^3b^3x^3}{c^8}-\frac{a^5bx^3}{c^8}+\frac{a^5b^3x^3}{c^{10}}$, &c.

$b+p=y$	$+y^3$	$+b^3+3b^2p+3bp^2+p^3$
	$+axy$	$+abx+axp$
	$+a^2y$	$+a^2b+a^2p$
	$-x^3$	$-x^3$
	$-2a^3$	$-2a^3$
$\frac{abx}{c^2}+q=p$	p^3	$-\frac{a^3b^3x^3}{c^6}$, &c.
	$+3bp^2$	$+\frac{3a^2b^3x^2}{c^4}-\frac{6ab^2x}{c^2}q$, &c.
	$+axp$	$-\frac{a^2bx^2}{c^2}+axq$
	$+c^2p$	$-abx+c^2q$
	$-x^3$	$-x^3$
	$+abx$	$+abx$

$c^2+ax-\frac{6ab^2x}{c^2}\Big)\,\frac{a^4bx^2}{c^4}+x^3+\frac{a^3b^3x^3}{c^6}\Big(\frac{a^4bx^2}{c^6}+\frac{x^3}{c^2}+\frac{a^3b^3x^3}{c^8}$, &c.

Here writing b in the Quote, I ſuppoſe $b+p=y$, and then for y I ſubſtitute as you ſee, whence proceeds p^3+3bp^2, &c. rejecting the Terms $b^3+a^2b-2a^3$ as being equal to nothing: for b is ſuppoſed to be a Root of this Equation $y^3+a^2y-2a^3=0$. Then the Terms $3b^2p+a^2p+abx$ give $\frac{-abx}{3b^2+a^2}$ to be ſet in the Quote, and $\frac{-abx}{a^2+3b^2}+q$ to be ſubſtituted for p. But for brevity's ſake I write cc for $aa+3bb$, yet with this caution, that $aa+3b^2$ may be reſtored whenſoever I perceive that the Terms may be abbreviated by it. When the Work is finiſh'd I aſſume ſome Number for a and reſolve this Equation $y^3+a^2y-2a^3=0$, as is ſhewn above concerning numeral Equations; and I ſubſtitute for b any one of its Roots if it has three Roots; or rather I free ſuch Equations from Spe-

cies

cies as far as I can, especially from the indefinite Species, and that after the manner before insinuated; and for the rest only, if any remain that cannot be expung'd, I put Numbers. Thus $y^3+a^2y-2a^3=0$ will be freed from a by dividing the Root by a, and it will become $y^3+y-2=0$, which Root being found, and multiply'd by a, must be substituted instead of b.

Hitherto I have suppos'd the indefinite Species to be little; but if it be suppos'd to approach nearly to a given Quantity, for that indefinitely small Difference I put some Species, and that being substituted I solve the Equation as before. Thus in the Equation $\frac{1}{5}y^5-\frac{1}{4}y^4+\frac{1}{3}y^3-\frac{1}{2}y^2+y+a-x=0$, it being known or suppos'd that x is nearly of the same Quantity as a, I suppose z to be their Difference; and then writing $a+z$ or $a-z$ for x, there will arise $\frac{1}{5}y^5-\frac{1}{4}y^4+\frac{1}{3}y^3-\frac{1}{2}y^2+y\pm z=0$, which may be resolv'd as before.

But if that Species be suppos'd to be indefinitely great, for its reciprocal which will therefore be indefinitely little, I put some Species, which being substituted, I proceed in the Resolution as before. Thus having $y^3+y^2+y-x^3=0$ where x is known or suppos'd to be very great, for the reciprocally little Quantity $\frac{1}{x}$ I put z, and substituting $\frac{1}{z}$ for x, there will arise $y^3+y^2+y-\frac{1}{z^3}=0$, whose Root is $y=\frac{1}{z}-\frac{1}{3}-\frac{2}{9}z+\frac{7}{81}z^2+\frac{5}{81}z^3$, &c. where x being restored, if you please, it will be $y=x-\frac{1}{3}-\frac{2}{9x}+\frac{7}{81x^2}+\frac{5}{81x^3}$, &c.

If it should happen that none of these Expedients should succeed to your desire, you may have recourse to another. Thus in the Equation $y^4-x^2y^2+xy^2+2y^2-2y+1=0$, whereas the first Term ought to be obtain'd from the Supposition that y^4

$+2y^2$

$+2y^2-2y+1=0$, which yet admits of no poſſible Root; you may try what can be done another way. As you may ſuppoſe that x is but little different from $+2$, or that $2+z=x$; then ſubſtituting $2+z$ for x, there will ariſe $y^4-z^2y^2-3zy^2-2y+1=0$, and the Quote will begin from $+1$; or if you ſuppoſe x to be indefinitely great, or $\frac{1}{x}=z$, you will have $y^4-\frac{y^2}{z^2}+\frac{y^2}{z}+2y^2-2y+1=0$, and $+z$ for the initial Term of the Quote. And thus by proceeding according to ſeveral Suppoſitions, you may extract and expreſs Roots after various ways.

If you ſhould deſire to find how many ſeveral ways this may be done, you muſt try what Quantities, when ſubſtituted for the indefinite Species in the propos'd Equation will make it diviſible by y, $+$ or $-$ ſome Quantity, or by y alone; which for Example ſake will happen in the Equation $y^3+axy+a^2y-x^3-2a^3=0$, by ſubſtituting $+a$, or $-a$, or $-2a$, or $\overline{-2a^3}|^{\frac{1}{3}}$, *&c.* inſtead of x: and thus you may conveniently ſuppoſe the Quantity x to differ little from $+a$, or $-a$, or $-2a$, or $\overline{-2a^3}|^{\frac{1}{3}}$; and thence you may extract the Root of the Equation propoſed after ſo many ways; and perhaps alſo after as many other ways, ſuppoſing theſe differences to be indefinitely great. Beſides if you take for the indefinite Quantity this or that of the Species which expreſs the Root, you may perhaps obtain your deſire after ſome other different ways: and farther ſtill by ſubſtituting any fictitious Value for the indefinite Species, ſuch as $az+bz^2$, $\frac{a}{b+z}$, $\frac{a+cz}{b+z}$, *&c.* and then proceeding as before in the Equations that will reſult.

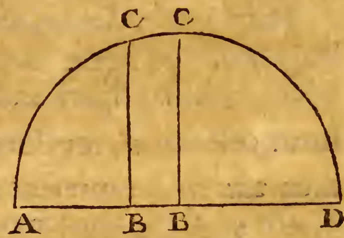
But now that the Truth of theſe Concluſions may be manifeſt, *i. e.* that the Quotes thus extracted and

and produced *ad libitum* approach ſo near to the Root of the Equation, as at laſt to differ from it by leſs than any aſſignable Quantity, and therefore when infinitely continued do not at all differ from it: You are to conſider that the Quantities in the left Hand Column of the right Hand Side of the Tables are the laſt Terms of the Equations whoſe Roots are p, q, r, s, *&c.* and that as they vaniſh, the Roots p, q, r, s, *&c.* *i. e.* the Differences between the Quote and the Root ſought, do likewiſe vaniſh at the ſame time; ſo that the Quote will not then differ from the true Root: Wherefore at the beginning of the Work if you ſee that the Terms in the ſaid Column will all deſtroy one another, you may conclude that the Quote ſo far extracted is the perfect Root of the Equation. But if it be otherwiſe, you will ſee however that the Terms in which the indefinitely ſmall Species is of few Dimenſions, that is, the greateſt Terms, are continually taken out of that Column, and that at laſt none will remain there, unleſs ſuch as are leſs than any given Quantity, and therefore not greater than nothing when the Work is continued *ad infinitum*. So that the Quote, when infinitely extracted, will at laſt be the true Root.

Laſtly, Although the Species which for the ſake of Perſpicuity I have hitherto ſuppos'd to be indefinitely little, ſhould however be ſuppos'd to be as great as you pleaſe, yet the Quotes will ſtill be true, though they may not converge ſo faſt to the true Root; this is manifeſt from the Analogy of the thing. But here the Limits of the Roots or the greateſt and leaſt Quantities come to be conſider'd; for theſe Properties are in common both to finite and infinite Equations. The Root in theſe is then greateſt or leaſt, when there is the greateſt or leaſt Difference between the Sums of the affirmative Terms and of the negative Terms; and is limited

limited when the indefinite Quantity, (which therefore not improperly I suppos'd to be ſmall,) cannot be taken greater, but that the Magnitude of the Root will immediately become infinite, that is, will become impoſſible.

To illuſtrate this, let ACD be a Semicircle deſcrib'd on the diameter AD and BC an ordinate. Make $AB = x$, $BC = y$, $AD = a$. Then $y = \sqrt{ax - x^2} = \sqrt{ax} - \frac{x}{2a}\sqrt{ax} - \frac{x^2}{8a^2}\sqrt{ax}$, &c. as before. Therefore BC or y then becomes greateſt, when the $\sqrt{ax}$ moſt exceeds all the terms $\frac{x}{2a}\sqrt{ax} + \frac{x^2}{8a^2}\sqrt{ax}$, &c. that is, when $x = \frac{1}{2}a$, but it will be terminated when $x = a$: for if we take x greater than a, the ſum of all the terms $-\frac{x}{2a}\sqrt{ax}$, $-\frac{x^2}{8a^2}\sqrt{ax}$, &c. will be infinite. There is another limit alſo when $x = 0$, by reaſon of the impoſſibility of the radical $\sqrt{-ax}$; to which terms or limits the limits of the ſemicircle A, B, and D, are correſpondent.



Tranſition to the Method of FLUXIONS.

And thus much for the Methods of Computation, of which I ſhall make frequent uſe in what follows. Now it remains, that for an illuſtration of the Analytic Art, I ſhould give ſome ſpecimens of Problems, eſpecially ſuch as the nature of Curves will ſupply. Now in order to this, I ſhall obſerve that all the difficulties hereof may be reduced to theſe two Problems only, which I ſhall propoſe, concerning a Space deſcrib'd by local Motion, any how accelerated or retarded.

I. *The length of the Space describ'd being continually (that is, at all times) given; to find the velocity of the motion at any time propos'd.*

II. *The velocity of the motion being continually given; to find the length of the Space describ'd at any time propos'd.*

Thus in the Equation $xx=y$, if y represents the length of the Space at any time describ'd, which (time) another Space x, by increasing with an uniform celerity $\dot{x}$, measures and exhibits as describ'd: then $2\dot{x}x$ will represent the celerity, by which the Space y at the same moment of time proceeds to be describ'd, and contrariwise. And hence it is, that in what follows I consider things as generated by continual Increase, after the manner of a Space, which a thing or point in motion describes.

But since we do not consider the time here, any farther than as it is expounded and measured by an equable local motion; and besides whereas things only of the same kind can be compar'd together, and also their velocities of increase and decrease: therefore in what follows I shall have no regard to time formally consider'd, but shall suppose some one of the quantities propos'd, being of the same kind, to be increas'd by an equable Fluxion, to which the rest may be refer'd, as it were to time; and therefore by way of analogy it may not improperly receive the name of Time. Whenever therefore the word *Time*, occurs in what follows, (which for the sake of perspicuity and distinction I have sometimes used,) by that word I would not have it understood as if I meant Time in its formal acceptation, but only that other quantity, by the equable increase or fluxion whereof, Time is expounded and measured.

Now

Now thoſe quantities which I conſider as gradually and indefinitely increaſing, I ſhall hereafter call *Fluents*, or *flowing Quantities*, and ſhall repreſent them by the final letters of the alphabet v, x, y, and z; that I may diſtinguiſh them from other quantities, which in equations may be conſidered as known and determinate, and which therefore are repreſented by the initial letters a, b, c, &c. And the velocities by which every Fluent is increaſed by its generating motion (which I may call *Fluxions*, or ſimply Velocities, or Celerities,) I ſhall repreſent by the ſame letters pointed thus, $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$; that is, for the celerity of the quantity v I ſhall put $\dot{v}$, and ſo for the celerities of the other Quantities x, y, and z, I ſhall put $\dot{x}$, $\dot{y}$, and $\dot{z}$, reſpectively. Theſe things being premis'd, I ſhall now forthwith proceed to the matter in hand; and firſt I ſhall give the ſolution of the two Problems juſt now propos'd.

PROBLEM I.

The Relation of the flowing Quantities to one another being given, to determine the Relation of their Velocities.

SOLUTION. Diſpoſe the equation, by which the given Relation is expreſs'd, according to the dimenſions of ſome one of its flowing Quantities, ſuppoſe x, and multiply its terms by any arithmetical progreſſion, and then by $\frac{\dot{x}}{x}$; and perform this operation ſeparately for every one of the flowing Quantities. Then make the ſum of all the products equal to nothing, and you will have the equation required.

EXAMPLE 1. If the relation of the flowing quantities x and y be $x^3 - ax^2 + axy - y^3 = 0$; first dispose the terms according to the dimensions of x, and then according to y, and multiply them in the following manner.

Mult. $x^3 - ax^2 + axy - y^3$	$-y^3 + axy \begin{matrix} -ax^2 \\ +x^3 \end{matrix}$
by $\frac{3\dot{x}}{x} \cdot \frac{2\dot{x}}{x} \cdot \frac{\dot{x}}{x} \cdot 0$	$\frac{3\dot{y}}{y} \cdot \frac{\dot{y}}{y} \cdot 0$
makes $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y$ *	$-3\dot{y}y^2 + a\dot{y}x$ *

the sum of the products is $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$, which equation gives the relation between the Fluxions $\dot{x}$ and $\dot{y}$. For if you take x at pleasure, the equation $x^3 - ax^2 + axy - y^3 = 0$ will give y; which being determin'd, it will be $\dot{x} : \dot{y} :: 3y^2 - ax : 3x^2 - 2ax + ay$.

Ex. 2. If the relation of the quantities x, y, and z, be exprefs'd by the equation $2y^3 + x^2y - 2cyz + 3yz^2 - z^3 = 0$,

Mult. $2y^3 + \begin{matrix} x^2 \\ -2cz \\ +3z^2 \end{matrix} \times y - z^3$	$yx^2 + \begin{matrix} 2y^3 \\ -2cyz \\ +3yz \\ -z^3 \end{matrix}$	$-z^3 + 3yz^2 - 2cyz + \begin{matrix} x^2y \\ +2y^3 \end{matrix}$
by $\frac{2\dot{y}}{y} \cdot 0 \cdot -\frac{\dot{y}}{y}$	$\frac{2\dot{x}}{x} \cdot 0$	$\frac{3\dot{z}}{z} \cdot \frac{2\dot{z}}{z} \cdot \frac{\dot{z}}{z} \cdot 0$
makes $4\dot{y}y^2$ * $+ \frac{\dot{y}z^3}{y}$	$2\dot{x}xy$ *	$-3\dot{z}z^2 + 6\dot{z}zy - 2c\dot{z}y$ *

Wherefore the relation of the celerities of flowing, or of the Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$, is $4\dot{y}y^2 + \frac{\dot{y}z^3}{y} + 2\dot{x}xy - 3\dot{z}z^2 + 6\dot{z}zy - 2c\dot{z}y = 0$.

But

But since there are here three flowing quantities x, y, and z, another equation ought also to be given, by which the relation among them, as also among their Fluxions, may be entirely determined. As if it were suppos'd that $x+y-z=0$. From hence another relation among the Fluxions $\dot{x}+\dot{y}-\dot{z}=0$ would be found by this rule. Now compare these with the foregoing equations, by expunging any one of the three Quantities, and also any one of the Fluxions, and then you will obtain an equation which will entirely determine the relation of the rest.

In the equation propos'd, whenever there are complex fractions or surd quantities, I put so many letters for each, and supposing them to represent flowing quantities, I work as before. Afterwards I suppress and exterminate the assum'd letters, as you see done here.

Ex. 3. If the relation of the quantities x and y be $yy-aa-x\sqrt{aa-xx}=0$, for $x\sqrt{a^2-x^2}$ I write z, and thence I have the two equations $yy-aa-z=0$, and $a^2x^2-x^4-z^2=0$, of which the first will give $2\dot{y}y-\dot{z}=0$ as before, for the relation of the celerities $\dot{y}$ and $\dot{z}$, and the latter will give $2a^2\dot{x}x-4\dot{x}x^3-2\dot{z}z=0$, or $\frac{a^2\dot{x}x-2\dot{x}x^3}{z}=\dot{z}$, for the relation of the celerities $\dot{x}$ and $\dot{z}$. Now $\dot{z}$ being expunged, it will be $2\dot{y}y\frac{-a^2\dot{x}x+2\dot{x}x^3}{z}=0$, and then restoring $x\sqrt{a^2-x^2}$ for z, we shall have $2\dot{y}y\frac{-a^2\dot{x}+2\dot{x}x^2}{\sqrt{a^2-x^2}}=0$, for the relation between $\dot{x}$ and $\dot{y}$ as was required.

Ex. 4.

Ex. 4. If $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay+xx} = 0$, expreſſes the relation that is between x and y: I make $\frac{by^3}{a+y} = z$, and $xx\sqrt{ay+x^2} = v$, from whence I ſhall have the three equations $x^3 - ay^2 + z - v = 0$, $az + yz - by^3 = 0$, and $ax^4y + x^6 - vv = 0$. The firſt gives $3\dot{x}x^2 - 2a\dot{y}y + \dot{z} - \dot{v} = 0$, the ſecond gives $a\dot{z} + \dot{z}y + \dot{y}z - 3b\dot{y}y^2 = 0$, and the third gives $4a\dot{x}x^3y + 6\dot{x}x^5 + a\dot{y}x^4 - 2\dot{v}v = 0$, for the relations of the velocities $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$; but the values of $\dot{z}$ and $\dot{v}$, found by the ſecond and third equations, (*i. e.* $\frac{3b\dot{y}y^2 - \dot{y}z}{a+y}$ for $\dot{z}$, and $\frac{4a\dot{x}x^3y + 6\dot{x}x^5 + a\dot{y}x^4}{2v}$ for $\dot{v}$) I ſubſtitute in the firſt equation, and there ariſes $3\dot{x}x^2 - 2a\dot{y}y + \frac{3b\dot{y}y^2 - \dot{y}z}{a+y} - \frac{4a\dot{x}x^3y - 6\dot{x}x^5 - a\dot{y}x^4}{2v} = 0$; then inſtead of z and v, reſtoring their values $\frac{by^3}{a+y}$ and $xx\sqrt{ay+x^2}$, there will ariſe the equation ſought

$$3\dot{x}x^2 - 2a\dot{y}y \frac{+3ab\dot{y}y^2 + 2b\dot{y}y^3}{aa+2ay+yy} - \frac{4a\dot{x}xy - 6\dot{x}x^3 - a\dot{y}xx}{2\sqrt{ay+xx}} = 0,$$

by which the relation of the velocities $\dot{x}$ and $\dot{y}$ will be expreſſed.

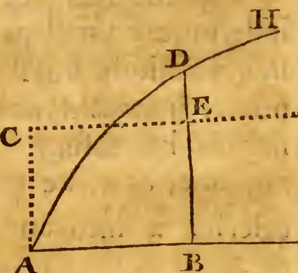
After what manner the operation is to be perform'd in other caſes, I believe is manifeſt from hence; as when in the equation propos'd there are found ſurd denominators, cubic radicals, radicals within radicals as $\sqrt{ax+\sqrt{aa-xx}}$, or any other complicate terms of the like kind.

Furthermore, although in the equation propos'd there ſhould be quantities involv'd, which cannot be determin'd or expreſs'd by any geometrical method, ſuch as curvilinear areas, or the lengths

lengths of curve-lines, yet the relations of their Fluxions may be found, as will appear from the following example.

Preparation for EXAMPLE 5.

Suppofe BD to be an ordinate at right angles to AB, and that ADH be any curve which is defin'd by the relation between AB and BD exhibited by an equation. Let AB be called x, and the area of the curve ADB, apply'd to unity, be called z. Then erect the perpendicular AC equal to unity, and through C draw CE parallel to AB and meeting BD in E. Then conceiving thefe two fuperficies ADB and ACEB to be generated by the motion of the right line BED; it is manifeft that their Fluxions (*i. e.* the Fluxions of the quantities $1 \times z$ and $1 \times x$, or of the quantities z and x) are to each other as the generating lines BD and BE. Therefore $\dot{z} : \dot{x} :: BD : BE$ or 1, and therefore $\dot{z} = \dot{x} \times BD$. And hence it is that z may be involv'd in any equation expreffing the relation between x and any other flowing quantity y; and yet the relation of the Fluxions $\dot{x}$ and $\dot{y}$ may however be difcover'd.



Ex. 5. As if the equation $zz + axz - y^4 = 0$ were propos'd to exprefs the relation between x and y, as alfo $\sqrt{ax - xx} = BD$ for determining a curve, which therefore will be a Circle. The equation $zz + axz - y^4 = 0$, as before, will give $2\dot{z}z + a\dot{z}x + a\dot{x}z - 4\dot{y}y^3 = 0$ for the relation of the celerities $\dot{x}$, $\dot{y}$, and $\dot{z}$. And therefore fince it is $\dot{z}$

$=\dot{x}$

$=\dot{x}\times BD$, or $=\dot{x}\sqrt{ax-xx}$, substitute this value instead of it, and there will arise the equation $\overline{2\dot{x}z+a\dot{x}x}\sqrt{ax-xx}+a\dot{x}z-4\dot{y}y^3=0$, which determines the relation of the celerities $\dot{x}$ and $\dot{y}$.

DEMONSTRATION *of the Solution.*

The Moments of flowing quantities (*i. e.* their indefinitely small parts, by the accession of which, in indefinitely small portions of time they are continually increas'd) are as the velocities of their flowing or increasing. Wherefore if the moment of any one, as x, be represented by the product of its celerity $\dot{x}$ into an indefinitely small quantity o, (*i. e.* by $\dot{x}o$,) the moments of the others v, y, and z, will be represented by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$; because $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$, are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$. Now since the moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little accessions of the flowing quantities x and y, by which those quantities are increased through the several indefinitely small intervals of time; it follows that those quantities x and y after any indefinitely small interval of time, become $x+\dot{x}o$ and $y+\dot{y}o$: and therefore the equation which at all times indifferently expresses the relation of the flowing quantities, will as well express the relation between $x+\dot{x}o$ and $y+\dot{y}o$, as between x and y: so that $x+\dot{x}o$ and $y+\dot{y}o$, may be substituted in the same equation for those quantities, instead of x and y.

Therefore let any equation $x^3-ax^2+axy-y^5=0$ be given, and substitute $x+\dot{x}o$ for x, and $y+\dot{y}o$ for y, and there will arise

$$x^3 + 3\dot{x}ox^2 + 3\dot{x}^2oox + \dot{x}^3o^3$$
$$-ax^2 - 2a\dot{x}ox - a\dot{x}^2oo$$
$$+axy + a\dot{x}oy + a\dot{y}ox + a\dot{x}\dot{y}oo$$
$$-y^3 - 3\dot{y}oy^2 - 3\dot{y}^2ooy - \dot{y}^3o^3 = 0.$$

Now by ſuppoſition $x^3 - ax^2 + axy - y^3 = 0$; which therefore being expung'd, and the remaining terms divided by o, there will remain $3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3oo - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + a\dot{y}x + a\dot{x}\dot{y}o - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3oo = 0$. But whereas o is ſuppos'd to be indefinitely little, that it may repreſent the moments of quantities, conſequently the terms that are multiplied by it, will be nothing in reſpect of the reſt: therefore I reject them, and there remains $3x^2\dot{x} - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$, as above in Example 1.

Here it may be obſerved, that the terms which are not multiplied by o will always vaniſh; as alſo thoſe terms that are multiplied by more than one dimenſion of o; and that the reſt of the terms being divided by o, will always acquire the form that they ought to have by the foregoing rule. Q. E. D.

This being done the other things inculcated in the rule will eaſily follow. As that in the propos'd equation, ſeveral flowing quantities may be involv'd; and that the terms may be multiply'd, not only by the number of the dimenſions of the flowing quantities, but alſo by any other arithmetick progreſſion, ſo that in the operation there may be the ſame difference of the terms according to any of the flowing quantities, and the progreſſion be diſpos'd according to ſome order of the dimenſions of each of them. Theſe things being allow'd, what is taught beſides in Examples 3, 4, and 5, will be plain enough of itſelf.

F PROBLEM

PROBLEM II.

An Equation being propos'd including the Fluxions of Quantities, to find the Relation of those Quantities to one another.

A particular Solution.

As this problem is the converse of the foregoing, it must be solv'd by proceeding in a contrary manner; that is, the terms multiply'd by $\dot{x}$ being dispos'd according to the dimensions of x, they must be divided by $\frac{\dot{x}}{x}$, and then by the number of their dimensions, or by some other arithmetical progression. Then the same work must be repeated with the terms multiplied by $\dot{v}$, $\dot{y}$, and $\dot{z}$, and the sum resulting must be put equal to nothing, rejecting the terms that are redundant.

Example. Let the equation propos'd be $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$, the operation will be after this manner.

Divide $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y$	Divide $-3\dot{y}y^2 \; * \; + a\dot{y}x$
by $\frac{\dot{x}}{x} = 3x^3 - 2ax^2 + ayx$	by $\frac{\dot{y}}{y} = -3y^3 \; * \; + axy$
div. by 3 . 2 . 1	div. by 3 . 2 . 1
Quote $x^3 \quad -ax^2 + ayx$	Quote $-y^3 \; * \; + axy$

Therefore the sum $x^3 - ax^2 + axy - y^3 = 0$ will be the required relation of the Quantities x and y. Where 'tis to be observ'd, that tho' the term axy occurs twice, yet I do not put it twice in the sum $x^3 - ax^2 + axy - y^3 = 0$; but I reject the redundant

dant

dant term. And ſo whenever any term occurs twice, or oftner, (as in caſes when there are ſeveral flowing quantities concerned,) it muſt be wrote only once in the ſum of the terms.

There are ſome other circumſtances to be obſerv'd, which I ſhall leave to the ſagacity of the Artiſt; for it would be needleſs to dwell long upon this matter, ſince the Problem cannot always be ſolved by this artifice. I ſhall add however, that after the relation of the Fluents is obtained by this method, if we can return, by PROB. I. to the propos'd equation involving the Fluxions, then the work is right, otherwiſe not. Thus, in the example propos'd, after I have found the equation $x^3 - ax^2 + axy - y^3 = 0$; if from thence I ſeek the relation of the Fluxions $\dot{x}$ and $\dot{y}$ by the firſt Problem, I ſhall arrive to the propos'd equation $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$: whence it is plain that the equation $x^3 - ax^2 + axy - y^3 = 0$ is rightly found. But if the equation $\dot{x}x - \dot{x}y + a\dot{y} = 0$ were propos'd, by the preſcrib'd method I ſhould obtain this $\frac{1}{2}x^2 - xy + ay = 0$ for the relation between x and y; which concluſion would be erroneous, ſince by PROB. I. the equation $\dot{x}x - \dot{x}y - \dot{y}x + a\dot{y} = 0$ would be produced, which is different from the former equation. Having therefore propos'd this in a perfunctory manner, I ſhall now undertake the general Solution.

Preparation for the general Rule.

Firſt it muſt be obſerved, that in the propos'd equation, the ſymbols of the Fluxions (ſince they are quantities of a different kind from the quantities of which they are the Fluxions) ought to aſcend in every term to the ſame number of dimenſions; and when it happens otherwiſe, another Fluxion of ſome flowing quantity muſt be underſtood to be unity, by which the lower terms are continually to be multiplied, till the ſymbols of the Fluxions ariſe to the ſame number of dimenſions in all the terms. As if the equation $\dot{x}+\dot{x}\dot{y}x-axx=0$ were propos'd, the Fluxion $\dot{z}$ of ſome third flowing quantity z muſt be underſtood to be unity, by which the firſt term $\dot{x}$ muſt be multiplied once, and the laſt axx twice; that the Fluxions in them may aſcend to as many dimenſions, as in the ſecond term $\dot{x}\dot{y}x$; as if the propos'd equation had been deriv'd from this $\dot{x}\dot{z}+\dot{x}\dot{y}x-a\dot{z}\dot{z}x^2=0$ by putting $\dot{z}=1$. And thus in the equation $\dot{y}x=yy$, I ought to imagine $\dot{x}$ to be unity, by which the term yy is to be multiplied.

Now equations, which have only two flowing quantities, that every where riſe to the ſame number of dimenſions, may always be reduced to ſuch a form, as that on one ſide may be had the ratio of the Fluxions, (as $\frac{\dot{y}}{\dot{x}}$, or $\frac{\dot{x}}{\dot{y}}$, or $\frac{\dot{z}}{\dot{x}}$, *&c.*) and on the other ſide the value of that ratio expreſſed by ſimple algebraic terms: as you may ſee here $\frac{\dot{y}}{\dot{x}}=2+2x-y$: and when the foregoing particular ſolution will not take place, it is requir'd that I ſhould bring the equation to this form.

When in the value of that Fluxion any term is denominated by a compound equation, or a radical, or if that Fluxion be the root of an affected equation, the reduction muſt be perform'd either by diviſion or by extraction of roots, or by the reſolution of an affected equation as has been before ſhewn.

So

So if the equation $\dot{y}a - \dot{y}x - \dot{x}a + \dot{x}x - \dot{x}y = 0$ were propos'd. First by reduction, this becomes $\frac{\dot{y}}{\dot{x}} = 1 + \frac{y}{a-x}$; or $\frac{\dot{x}}{\dot{y}} = \frac{a-x}{a-x+y}$: And in the first case, if I reduce the term $\frac{y}{a-x}$, denominated by the compound quantity $a-x$, to an infinite series of simple terms $\frac{y}{x} + \frac{xv}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$, *&c.* by dividing the numerator y by the denominator $a-x$, I shall have $\frac{\dot{y}}{\dot{x}} = 1 + \frac{y}{x} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$, *&c.* by the help of which the relation between x and y may be determined. So the equation $\dot{y}\dot{y} = \dot{x}\dot{y} + \dot{x}\dot{x}xx$ being given, or $\frac{\dot{y}\dot{y}}{\dot{x}\dot{x}} = \frac{\dot{y}}{\dot{x}} + xx$, and by a farther reduction $\frac{\dot{y}}{\dot{x}} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + xx}$, I extract the square root out of the terms $\frac{1}{4} + xx$, and obtain the infinite series $\frac{1}{2} + x^2 - x^4 + 2x^6 + 5x^8 + 14x^{10}$, *&c.* which if I substitute for $\sqrt{\frac{1}{4} + xx}$, I shall have $\frac{\dot{y}}{\dot{x}} = 1 + x^2 - x^4 + 2x^6 - 5x^8$, *&c.* or $\frac{\dot{y}}{\dot{x}} = -x^2 + x^4 - 2x^6 + 5x^8$, *&c.* according as the $\sqrt{\frac{1}{4} + xx}$ is either added to $\frac{1}{2}$, or substracted from it. And thus if the equation $\dot{y}^3 + ax\dot{x}^2\dot{y} + a^2\dot{x}^2\dot{y} - x^3\dot{x}^3 - 2\dot{x}^3a^3 = 0$ were propos'd, or $\frac{\dot{y}^3}{\dot{x}^3} + ax\frac{\dot{y}}{\dot{x}} + a^2\frac{\dot{y}}{\dot{x}} - x^3 - 2a^3 = 0$, I extract the root of this affected cubick equation, and there arises $\frac{\dot{y}}{\dot{x}} = a - \frac{x}{4} + \frac{xx}{64a} +$

$131x^3$

$\frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$, &c. as may be ſeen before.

But here it may be obſerv'd, that I look upon thoſe terms only as compounded, which are compounded in reſpect of flowing quantities; for I eſteem thoſe as ſimple quantities which are compounded only in reſpect of given quantities, ſince they may be reduced to ſimple quantities by ſuppoſing them equal to other quantities. Thus I conſider the quantities $\frac{ax+bx}{c}$, $\frac{x}{a+b}$, $\frac{bcc}{ax+bx}$, $\frac{b^4}{ax^2+bx^2}$, $\sqrt{ax+bx}$, &c. as ſimple quantities, becauſe they may all be reduced to the ſimple quantities $\frac{ex}{c}$, $\frac{x}{e}$, $\frac{bc^2}{ex}$, $\frac{b^4}{ex^2}$, $\sqrt{ex}$, (or $e^{\frac{1}{2}}x^{\frac{1}{2}}$,) &c. by ſuppoſing $a+b=e$.

Moreover, that the flowing quantities may the more eaſily be diſtinguiſhed from one another, the Fluxion that is put in the numerator of the fraction or the Antecedent of the ratio may not improperly be called the *Relate Quantity*, and the other in the denominator, to which it is compared, the *Correlate*. Alſo the Flowing Quantities may be diſtinguiſhed by the ſame names reſpectively. And for the better underſtanding of what follows, you may conceive that the correlate quantity is *time*, or rather any other quantity that flows equably, by which *time* is expounded and meaſured; and that other, or the relate quantity, is *ſpace*, which the moving thing or point any how accelerated or retarded deſcribes in that *time*; and that it is the intention of the Problem, that from the velocity of the motion being given at every inſtant of *time*, the *ſpace* deſcrib'd in the whole *time* may be determin'd.

But in reſpect of this Problem, equations may be diſtinguiſhed into three orders.

1. Thoſe

I. Thofe in which two Fluxions of quantities and only one of their flowing quantities are involv'd.

II. Thofe in which the two flowing quantities are involv'd together with their Fluxions.

III. Thofe in which the Fluxions of more than two quantities are involv'd. ——With thefe premifes I fhall attempt the folution of the Problem according to thefe three cafes.

Solution of Cafe I.

Suppofe the flowing quantity, which alone is contain'd in the equation, to be the correlate: and the equation being accordingly difpofed, (*i. e.* by making on one fide to be only the ratio of the Fluxions; and on the other fide the value of this ratio in fimple terms,) firft multiply the value of the ratio of the Fluxions by the correlate quantity, then divide each of its terms by the number of dimenfions with which that quantity is there affected, and what arifes will be equivalent to the other flowing quantity.

So propofing the equation $\dot{y}\dot{y} = \dot{x}\dot{y} + \dot{x}\dot{x}xx$, I fuppofe x to be the correlate quantity, and the equation being accordingly reduced, we fhall have $\frac{\dot{y}}{\dot{x}} = 1 + x^2 - x^4 + 2x^6$, &c. now multiply the value of $\frac{\dot{y}}{\dot{x}}$ into x, and there arifes $x + x^3 - x^5 + 2x^7$, &c. which terms I divide feverally by their number of dimenfions, and the refult $x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7$, &c. I put $= y$, and by this equation will be defined the relation between x and y as was required.

Let the equation be $\frac{\dot{y}}{\dot{x}} = a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^3}{512a^2}$ &c. there will arife $y = ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2}$, &c.

&c. for determining the relation between x and y. Thus the equation $\frac{\dot{y}}{\dot{x}} = \frac{1}{x^3} - \frac{1}{x^2} + \frac{a}{x^{\frac{1}{2}}} - x^{\frac{1}{2}} + x^{\frac{3}{2}}$ &c. gives $y = \frac{1}{2}x^2 + \frac{1}{x} + 2ax^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}$, &c. for multiplying the value of $\frac{\dot{y}}{\dot{x}}$ into x it becomes $\frac{1}{xx} - \frac{1}{x} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}}$, &c. or $x^{-\frac{1}{2}} - x^{-1} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + a^{\frac{5}{2}}$, &c. which terms being divided by the number of dimensions, the value of y will arise as before.

After the same manner the equation $\frac{\dot{y}}{\dot{x}} = \frac{2b^2c}{\sqrt{ay^3}} + \frac{3y^2}{a+b} + \sqrt{by+cy}$, gives $x = -\frac{4b^2c}{\sqrt{ay}} + \frac{y^3}{a+b} + \frac{2}{3}\sqrt{by^3+cy^3}$; for the value of $\frac{\dot{x}}{\dot{y}}$ multiplied by y, there arises $\frac{2b^2c}{\sqrt{ay}} + \frac{3y^3}{a+b} + \sqrt{by^3+cy^3}$, or $2b^2ca^{-\frac{1}{2}}y^{-\frac{1}{2}} + \frac{3}{a+b}y^3 + \sqrt{b+c} \times y^{\frac{3}{2}}$, and thence the value of x results by dividing by the number of the dimensions of each term.

The equation $\frac{\dot{y}}{\dot{x}} = z^{\frac{2}{3}}$, gives $y = \frac{3}{5}z^{\frac{5}{3}}$. And $\frac{\dot{y}}{\dot{x}} = \frac{ab}{cx^{\frac{1}{3}}}$, gives $y = \frac{3abx^{\frac{2}{3}}}{2c}$.

But the equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$, gives $y\ \frac{a}{0}$; for $\frac{a}{x} \times x$ makes a, which being divided by the number of dimensions which is 0, there arises $\frac{a}{0}$, an infinite quantity, for the value of y.

When-

Whenever a like term ſhall occur in the value of $\frac{\dot{y}}{\dot{x}}$ whoſe denominator involves the Correlate Quantity of one dimenſion only, inſtead of the Correlate Quantity ſubſtitute the ſum or the difference between the ſame and ſome other given quantity to be aſſumed at pleaſure; for there will be the ſame relation of the Fluxions of the Fluents in the equation ſo produced, as of the other equation at firſt propos'd; and the infinite Relate Quantity by this means will be diminiſhed by an infinite part of itſelf, and will become finite, but yet conſiſting of terms infinite in number.

Therefore the equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$ being propoſed, if for x I write $b+x$, aſſuming the quantity b at pleaſure, there will ariſe $\frac{\dot{y}}{\dot{x}} = \frac{a}{b+x}$, and by diviſion $\frac{\dot{y}}{\dot{x}} = \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^4}{b^4}$, &c. And now the foregoing rule will give $y = \frac{ax}{b} - \frac{ax^2}{2b^3} + \frac{ax^3}{3b^3} - \frac{ax^4}{4b^4}$, &c. for the relation between x and y.

So in the equation $\frac{\dot{y}}{\dot{x}} = \frac{2}{x} + 3 - xx$, if, becauſe of the term $\frac{2}{x}$, I write $1+x$ for x, there will ariſe $\frac{\dot{y}}{\dot{x}} = \frac{2}{1+x} + 2 - 2x - xx$: then reducing the term $\frac{2}{1+x}$ into an infinite ſeries $2 - 2x + 2x^2 - 2x^3 + 2x^4$, &c. we ſhall have $\frac{\dot{y}}{\dot{x}} = 4 - 4x + x^2 - 2x^3 + 2x^4$ &c. and then according to the rule $y =$

$4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{2}{5}x^5$, &c. for the relation of x and y.

And thus if the equation $\frac{\dot{y}}{\dot{x}} = x^{-\frac{1}{2}} + x^{-1} - x^{\frac{1}{2}}$ were propofed, becaufe I here obferve the term x^{-1} (or $\frac{1}{x}$) to be found, I tranfmute x by fubftituting $1-x$ for it, and there arifes $\frac{\dot{y}}{\dot{x}} = \frac{1}{\sqrt{1-x}} + \frac{1}{1-x} - \sqrt{1-x}$. Now the term $\frac{1}{1-x}$ produces $1 + x + x^2 + x^3$, &c. and the term $\sqrt{1-x}$ is equivalent to $1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$, and therefore $\frac{1}{\sqrt{1-x}}$ or $\frac{1}{1 - \frac{1}{2}x - \frac{1}{8}x^2}$, &c. is the fame as $1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3$, &c. fo that when thefe values are fubftituted we fhall have $\frac{\dot{y}}{\dot{x}} = 1 + 2x + \frac{3}{2}x^2 + \frac{11}{8}x^3$, &c. and then by the rule $y = x + x^2 + \frac{1}{2}x^3 + \frac{11}{32}x^4$, &c. And fo of others.

Alfo in other cafes the equation may fometimes be conveniently reduced by fuch a tranfmutation of the flowing quantity. As if this equation were propofed $\frac{\dot{y}}{\dot{x}} = \frac{c^2x}{c^3 - 3c^2x + 3cx^2 - x^3}$, inftead of x I write $c-x$, and then I fhall have $\frac{\dot{y}}{\dot{x}} = \frac{c^3 - c^2x}{x^3}$ or $\frac{c^3}{x^3} - \frac{c^2}{x^2}$, and then by the rule $y = -\frac{c^3}{2xx} + \frac{c^2}{x}$. But the ufe of fuch tranfmutations will appear more plainly in what follows.

Solution of Cafe II.

Preparation. And fo much for equations that involve only one Fluent, but when each of them are

are found in the equation; first they must be reduced to the form prescribed, by making the ratio of the Fluxions on one side equal to any aggregate of simple terms on the other. And further, if in the equation so reduced, there be any fractions denominated by the flowing quantity, they must be freed from those denominators by the abovementioned transmutation of the flowing quantities.

Thus the equation $\dot{y}ax - \dot{x}xy - aa\dot{x} = 0$ being proposed, or $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{x}$; by reason of the term $\frac{a}{x}$, I assume b at pleasure, and for x I write either $b+x$, $b-x$ or $x-b$. As if for instance I should write $b+x$; it will become $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{b+x}$; and then the term $\frac{a}{b+x}$ being converted by division into an infinite series, we shall have $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4}$, &c.

In like manner the equation $\frac{\dot{y}}{\dot{x}} = 3y - 2x + \frac{x}{y} - \frac{2y}{xx}$ being propos'd; if, by reason of the terms $\frac{x}{y}$ and $\frac{2y}{xx}$, I write $1-y$ for y, and $1-x$ for x; there will arise $\frac{\dot{y}}{\dot{x}} = 1 - 3y + 2x + \frac{1-x}{1-y} + \frac{2y-2}{1-2x+x^2}$. But the term $\frac{1-x}{1-y}$ by infinite division, gives $1 - x + y - xy + y^2 - xy^2 + y^3 - xy^3$, &c. And the term $\frac{2y-2}{1-2x+xx}$ by a like division gives

 2y—

$2y - 2 + 4xy - 4x + 6x^2y - 6x^2 + 8x^3y - 8x^3 + 10x^4y - 10x^4$, &c. therefore $\frac{\dot{y}}{\dot{x}} = -3x + 3xy + y^2 - xy^2 + y^3 - xy^3$, &c. $+ 6x^2y - 6x^2 + 8x^3y - 8x^3$ $10x^4y - 10x^4$, &c.

Rule. The equation being thus prepared, when the caſe requires it, diſpoſe the terms according to the dimenſions of the flowing quantities, in the following manner. Firſt ſelect thoſe that are not affected with the Relate Quantity; then thoſe that are affected by its leaſt dimenſions; and ſo on. In the next place diſpoſe the terms of each ſeries thus ſelected, into their ſeveral partitions, according to the dimenſions of the Correlate Quantity, writing thoſe in the firſt partition (*i. e.* ſuch as are not affected by the Relate Quantity) in a collateral order, proceeding towards the right hand, and the reſt in deſcending ſeries on the left hand column, as you ſee in the following Tables. The work being thus prepared, multiply the firſt or the leaſt of the terms in the firſt partition by the Correlate Quantity; then dividing that product by the number of its dimenſions, put the reſult in the quote for the initial term of the value of the Relate Quantity. Then ſubſtitute this value inſtead of the Relate Quantity into the terms of the equation that are placed in the left hand column; and from the next leaſt term you will obtain the ſecond term of the quote, by the ſame proceſs that you obtained the firſt. Thus repeating the operation, you may continue the quote as far as you pleaſe. But this will be plainer by an example or two.

EXAMPLE I. Let the equation $\frac{\dot{y}}{\dot{x}} = 1 - 3x + y + x^2 + xy$ be propoſed, whoſe terms $1 - 3x + x^2$ (which are not affected by the Relate Quantity y)

you

you see disposed collaterally in the uppermost partition; and the rest, y, and xy, in the left hand column. This done, first I multiply the initial

	$+1-3x+xx$
$+y$	$* + x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5$, &c.
$+xy$	$* \; * + xx - x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5 + \frac{1}{30}x^6$, &c.
Sum	$1 - 2x + xx - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{4}{30}x^5$, &c.
$y=$	$x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$, &c.

term, 1, into the Correlate Quantity, x, and it makes x; which being divided by the number of dimensions 1, I place in the quote above written; then substituting these terms instead of y in the marginal terms y and $+xy$, I have $+x$ and $+xx$, which I write overagainst them to the right hand. After which from the rest I take the least terms $-3x$ and $+x$, whose aggregate $-2x$ multiplied into x becomes $-2xx$, this divided by the number of its dimensions 2, gives $-xx$ for the second term of the value of y in the quote. In proceeding this term being likewise assumed to complete the value of the marginals $+y$ and $+xy$, there will arise $-xx$ and $-x^3$ to be added to the terms $+x$ and $+xx$, that were before inserted: which being done, I again assume the next least terms, $+xx$, $-xx$, and $+xx$, which I collect into one sum $+xx$, and thence I derive (as before) the third term $+\frac{1}{3}x^3$ to be put into the value of y. Again, taking this term $\frac{1}{3}x^3$ into the place of the marginals; from the next least terms, $+\frac{1}{3}x^3$ and $-x^3$ added together, I obtain $\frac{1}{6}x^4$ for the fourth

fourth term of the value of y. And so on *in infinitum.*

Ex. 2. In like manner if it were requir'd to determine the relation of x and y in this equation $\frac{\dot{y}}{\dot{x}} = 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$, &c. which series is supposed to proceed *ad infinitum.* I put 1 in the beginning, and the other terms in the left hand column, and then pursue the work according to the following table.

	$+1$
$+\frac{y}{a}$	$* \; \frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{2a^3} + \frac{x^4}{2a^4} + \frac{x^5}{2a^5}$, &c.
$+\frac{xy}{a^2}$	$* \; * \; \frac{x^2}{a^2} + \frac{x^3}{2a^3} + \frac{x^4}{2a^4} + \frac{x^5}{2a^5}$, &c.
$+\frac{x^2y}{a^3}$	$* \; * \; * \; \frac{x^3}{a^3} + \frac{x^4}{2a^4} + \frac{x^5}{2a^5}$, &c.
$+\frac{x^3y}{a^4}$	$* \; * \; * \; * \; \frac{x^4}{a^4} + \frac{x^5}{2a^5}$, &c.
$+\frac{x^4y}{a^5}$	$* \; * \; * \; * \; * \; + \frac{x^5}{a^5}$, &c.
The Sum	$1 + \frac{x}{a} + \frac{3x^2}{2a^2} + \frac{2x^3}{a^3} + \frac{5x^4}{2a^4} + \frac{3x^5}{a^5}$, &c.
$y=$	$x + \frac{x^2}{2a} + \frac{x^3}{2a^2} + \frac{x^4}{2a^3} + \frac{x^5}{2a^4} + \frac{x^6}{2a^5}$, &c.

As I have proposed to extract the value of y as far as six dimensions only of x, for that reason I omit all the terms in the operation which I foresee will contribute

contribute nothing to my purpose, as is intimated by the mark, &c. subjoined to the series that are cut off.

Ex. 3. In like manner, if this equation were proposed, $\frac{\dot{y}}{\dot{x}} = -3x + 3xy + y^2 - xy^2 + y^3 - xy^3 + y^4 - xy^4$, &c. $+ 6x^2y - 6x^2 + 8x^3y - 8x^3 + 10x^4y - 10x^4$, &c. and it is intended to extract the value of y as far as seven dimensions of x. I place the terms in order according to the following table; and

	$-3x$	$-6x^2$	$-8x^3$	$-10x^4$	$-12x^5$	$-14x^6$	&c.
$+3xy$	*	*	$-\frac{9}{2}x^3$	$-6x^4$	$-\frac{75}{8}x^5$	$-\frac{273}{20}x^6$	&c.
$+6x^2y$	*	*	*	$-9x^4$	$-12x^5$	$-\frac{75}{4}x^6$	&c.
$+8x^3y$	*	*	*	*	$-12x^5$	$-16x^6$	&c.
$+10x^4y$	*	*	*	*	*	$-15x^6$	&c.
&c.							
$+y^2$	*	*	*	$+\frac{9}{4}x^4$	$+6x^5$	$+\frac{107}{8}x^6$	&c.
$-xy^2$	*	*	*	*	$-\frac{9}{4}x^5$	$-6x^6$	&c.
&c.							
$+y^3$	*	*	*	*	*	$-\frac{27}{8}x^6$	&c.
The Sum	$-3x$	$-6x^2$	$-\frac{25}{2}x^3$	$-\frac{91}{4}x^4$	$-\frac{333}{8}x^5$	$-\frac{367}{5}x^6$	&c.
$y =$	$-\frac{3}{2}x^2$	$-2x^3$	$-\frac{25}{8}x^4$	$-\frac{91}{20}x^5$	$-\frac{111}{16}x^6$	$-\frac{367}{35}x^7$	&c.
$y^2 =$	$+\frac{9}{4}x^4$	$+6x^5$	$+\frac{107}{8}x^6$	&c.			
$y^3 =$	$-\frac{27}{8}x^6$	&c.					

work as before, only with this exception, that since in the left hand column y is not only of one, but also of two and of three dimensions, (or of more than three, if I intended to produce the value of y beyond the degree of x^7;) I subjoin the square and cube of the value of y so far gradually produced, that when they are substituted by degrees to the

the right hand in the values of the marginals on the left, terms may arise of so many dimensions as I observe to be required for the following operation: And by this method there is produced at length $y = -\frac{3}{2}x^2 - 2x^3 - \frac{25}{8}x^4$, &c. which is the equation required. But whereas this value is negative, it appears that one of these quantities x or y decreases, while the other increases. And the same thing is also to be concluded, when one of the Fluxions is affirmative and the other negative.

Ex. 4. You may proceed in like manner to resolve the equation, when the Relate Quantity is affected with fractional dimensions. As if it were proposed to extract the value of x from this equation $\frac{\dot{x}}{\dot{y}} = \frac{1}{2}y - 4y^2 + 2yx^{\frac{1}{2}} - \frac{4}{5}x^2 + 7y^{\frac{5}{2}} + 2y^3$; in which x in the term $2yx^{\frac{1}{2}}$ (or $2y\sqrt{x}$) is affected with the fractional dimension $\frac{1}{2}$. From the value of x I derive by degrees the value of $x^{\frac{1}{2}}$, (*i. e.* by extracting its square root) as may be observed in the lower part of the table, that it may be inserted

	$+\frac{1}{2}y$	*	$-4y^2$	$+7y^{\frac{5}{2}}$	$+2y^3$		
$2yx^{\frac{1}{2}}$	*	*	$+y^2$	*	$-2y^3$	$+4y^{\frac{7}{2}}$	$-2y^4$, &c.
$-\frac{4}{5}x^2$	*	*	*	*	*	*	$-\frac{1}{20}y^4$, &c.
Sum	$+\frac{1}{2}y$	*	$-3y^2$	$+7y^{\frac{5}{2}}$	*	$+4y^{\frac{7}{2}}$	$-\frac{41}{20}y^4$, &c
$x =$	$+\frac{1}{4}y^2$	*	$-y^3$	$+2y^{\frac{7}{2}}$	*	$+\frac{8}{9}y^{\frac{9}{2}}$	$-\frac{41}{100}y^5$, &c.
$x^{\frac{1}{2}} =$	$+\frac{1}{2}y$	$-y^2$	$+2y^{\frac{5}{5}}$	$-y^3$, &c.			
$x^2 =$	$\frac{1}{16}y^4$, &c.						

and

and transferred by degrees into the value of the marginal term $2yx^{\frac{1}{2}}$, and ſo at laſt I ſhall have the equation $x = \frac{1}{4}y^2 - y^3 + 2y^{\frac{7}{2}} + \frac{8}{9}y^{\frac{9}{2}} - \frac{41}{100}y^5$, &c. by which x is expreſſed indefinitely in reſpect of y. And thus you may operate in any other caſe whatſoever.

I ſaid before that theſe operations may be performed an infinite variety of ways; this may be done if you aſſume at pleaſure, not only the initial quantity of the upper ſeries, but any other given quantity, for the firſt term of the quote, and then proceed as before. Thus in the firſt of the propoſed examples, if you aſſume 1 for the firſt term of the value of y, and ſubſtitute it for the value of y into the marginal terms $+y$ and $+xy$, and purſue the reſt of the operation as before, (a ſpecimen of which I havehere given) another value

	$+1 - 3x + xx$
$+y$	$+1 + 2x \quad * \quad + x^3 + \frac{1}{4}x^4$, &c.
$+xy$	$* + x + 2x^2 \quad * \quad + x^4$, &c.
The Sum	$+2 \quad * \quad + 3x^2 + x^3 + \frac{5}{4}x^4$, &c.
$y=$	$1 + 2x \quad * \quad + x^3 + \frac{1}{4}x^4 + \frac{1}{4}x^5$, &c.

of y will ariſe, *viz.* $1 + 2x + x^3 + \frac{1}{4}x^4$, &c. And thus another and another value may be produced, by aſſuming 2, or 3, or any other number for its

 firſt

first term. Or if you make use of any symbol, as a, to represent the first term indefinitely, by the same method of operation (which I shall here exhibit) you will find $y = a + x + ax - xx + axx$

	$+1 - 3x + xx$
$+y$	$+a + x \quad - xx \quad + \frac{1}{3}x^3$, &c.
	$+ax \quad + ax^2 \quad + \frac{2}{3}ax^3$, &c.
$+ay$	$* + ax \quad + x^2 \quad - x^3$, &c.
	$+ax^2 \quad + ax^3$, &c.
The Sum	$+1 - 2x \quad + x^2 \quad - \frac{3}{2}x^3$, &c.
	$+a + 2ax + 2ax^2 + \frac{5}{3}ax^3$, &c.
$= y$	$a + x \quad + x^2 \quad + \frac{1}{3}x^3 \quad - \frac{1}{6}x^4$, &c.
	$+ax \quad + ax^2 + \frac{2}{3}ax^3 + \frac{5}{12}ax^4$, &c.

$+ \frac{1}{3}x^3 + \frac{2}{3}ax^3$, &c. which being found for a, I may substitute 1, 2, 0, $\frac{1}{2}$, or any other number, and thereby obtain the relation between x and y an infinite variety of ways.

And it is to be observed, that when the quantity to be extracted is affected with a fractional dimension, (as you see in the fourth of the foregoing examples) then it is convenient to take unity or some other proper number for its first term: And indeed this is absolutely necessary, when to obtain the value of that fractional dimension the root cannot otherwise be extracted, because of the negative sign, as also when there are no terms to be disposed in the first or uppermost partition, from whence that initial term may be deduced.

And thus at last I have completed this most troublesome, and of all others most difficult Problem, when

when only two Flowing Quantities, together with their Fluxions, are comprehended in an equation. But besides this general method into which I have taken all the difficulties, there are others much shorter by which the work may often be eased in particular cases: to give some specimens of which *ex abundanti*, will not perhaps be disagreeable to the Reader.

If it happens that the quantity to be resolved has in some places negative dimensions, it is not therefore absolutely necessary that the equation should be reduced to another form. For thus the equation $\dot{y} = \frac{1}{y} - xx$ being proposed, where y is of one negative dimension; I might indeed reduce it to another form, as by writing $1+y$ for y: But the resolution will be more expedite, as you have it in the following table.

	$* \quad * - xx$
$\frac{1}{y}$	$1 - x + \frac{3}{2}xx$, &c.
The Sum	$1 - x + \frac{1}{2}xx$, &c.
$y =$	$1 + x - \frac{1}{2}xx + \frac{1}{6}x^3$, &c.
$\frac{1}{y} =$	$1 - x + \frac{3}{2}xx$, &c.

1. Here assuming 1 for the initial term of the value of y, I extract the rest of the terms as before, and in the mean time from thence, by degrees, I deduce

duce the value of $\frac{1}{y}$ by division, and insert it into the value of the marginal term.

2. Neither is it necessary that the dimension of the other flowing quantity should be always affirmative. For from the equation $\dot{y}=3+2y-\frac{yy}{x}$ without the prescribed reduction of the term $\frac{yy}{x}$ there will arise $y=3x-\frac{3}{2}xx+2x^3$, &c. And from the equation $\dot{y}=-y+\frac{1}{x}-\frac{1}{xx}$ the value of y will be found $y=\frac{1}{x}$, if the operation be performed after the manner of the following specimen.

	$-\frac{1}{xx} \quad +\frac{1}{x}$
$-y$	$* \quad -\frac{1}{x}$
The Sum	$-\frac{1}{xx} \quad *$
$y=$	$\frac{1}{x}$

Here we may observe by the way, that among the infinite manners by which any infinite equation may be resolved, it often happens that there are some that terminate at a finite value of the quantity to be extracted, as in the foregoing example. And these are not difficult to find if some symbol be assumed for the first term; for after the resolution is performed, then some proper value of that symbol may be given, which may render the whole finite.

3. Again, if the value of y is to be extracted from this equation $\dot{y}=\frac{y}{2x}+1-2x+\frac{1}{2}xx$, it may be done conveniently enough without any reduction

tion of the term $\frac{y}{2x}$; by ſuppoſing (after the manner of Analyſts) that to be given, which is required. Thus for the firſt term of the value of y, I put $2ex$, taking $2e$ for the numeral co-efficient which is yet unknown; and ſubſtituting $2ex$ inſtead of y into the marginal term, there ariſes e, which I write on the right hand, and the ſum $1+e$, will give $x+ex$ for the ſame firſt term of the value of y, which I had firſt repreſented by the term $2ex$; therefore I make $2ex=x+ex$, and thence deduce $e=1$: So that the firſt fictitious term $2ex$ of the value of y, is really $2x$. After the ſame manner I make uſe of the fictitious term $2fx^2$, to repreſent the ſecond term of the value of y, and thence at laſt derive $-\frac{2}{3}$ for the value of f; therefore that ſecond term is really $-\frac{4}{3}xx$. And ſo the fictitious co-efficient g in the third term will give $\frac{1}{10}$. And h in the fourth term will be o. Therefore ſince there are no other terms remaining, I conclude the work is finiſhed, and that the value is exactly $2x-\frac{4}{3}x^2+\frac{1}{5}x^3$. See the operation in the following table.

	$1-2x+\frac{1}{2}xx$
$+\frac{y}{2x}$	$e+fx+gxx+hx^3$
The Sum	$+1-2x+\frac{1}{2}xx$ $+e+fx+gxx+hx^3$
Hypothetically	$y=2ex+2fx^2+2gx^3+2hx^4$
Conſequentially	$y=+x-x^2+\frac{1}{6}x^3+\frac{1}{4}hx^4$ $+ex+\frac{1}{2}fx^2+\frac{1}{3}gx^3$
Real value	$y=2x-\frac{4}{3}x^2+\frac{1}{5}x^3.$

Much

Much after the ſame manner if it were $\dot{y} = \frac{3y}{4x}$. Suppoſe $y = ex^s$; where e denotes the unknown co-efficient, and s the number of dimenſions, which is alſo unknown: then ex^s being ſubſtituted for y, there will ariſe $\dot{y} = \frac{3ex^{s-1}}{4}$, and thence again $y = \frac{3ex^s}{4s}$; compare theſe two values of y, and you will find $\frac{3e}{4s} = e$, therefore $s = \frac{3}{4}$, and e will be indefinite.

4. Sometimes alſo the operation may be begun from the higheſt dimenſion of the equable quantity, and continually proceed to the lower powers. As if this Equation were given $\dot{y} = \frac{y}{xx} + \frac{1}{xx} + 3 + 2x - \frac{4}{x}$, and we would begin from the higheſt term $2x$; by diſpoſing the capital Series in any order contrary to the foregoing, there will ariſe at laſt $y = xx + 4x - \frac{1}{x}$, &c. as may be ſeen in the form of working here ſet down.

	$+2x+3-\frac{4}{x}+\frac{1}{xx}$
$+\frac{y}{xx}$	$*\ +1+\frac{4}{x}\ *\ -\frac{1}{x^3}+\frac{1}{2x^4}$, &c.
The Sum	$+2x+4\ *\ +\frac{1}{xx}-\frac{1}{x^3}+\frac{1}{2x^4}$, &c.
$y=$	$x^2+4x\ *\ -\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{6x^3}$, &c.

Here it may be obſerved by the way, that as the operation proceeded, I might have inſerted any

any given quantity between the terms $4x$ and $-\frac{1}{x}$ for the intermediate term that is deficient, and ſo the value of y might have been extracted an infinite variety of ways.

If there are beſides any fractional indices of the dimenſions of the Relate Quantity, they may be reduced to Integers, by ſuppoſing the ſaid quantity ſo affected by its fractional dimenſion to be equal to any third Fluent, and thence by ſubſtitution of that quantity, as alſo of its Fluxion ariſing from the fictitious equation, inſtead of the Relate Quantity and its Fluxion.

As if the equation $\dot{y}=3xy^{\frac{2}{3}}+y$ were propoſed, where the Relate Quantity is affected with the fractional index $\frac{2}{3}$. A Fluent z being aſſumed at pleaſure, ſuppoſe $y^{\frac{1}{3}}=z$, or $y=z^3$; then the relation of the Fluxions by PROB. I. will be $\dot{y}=3\dot{z}z^2$: Therefore ſubſtituting $3\dot{z}z^2$ for $\dot{y}$, as alſo z^3 for y, and z^2 for $y^{\frac{2}{3}}$, there will ariſe $3\dot{z}z^2=3xz^2+z^3$ or $\dot{z}=x+\frac{1}{3}z$; where z performs the office of the Relate Quantity. But after the value of z is extracted as $z=\frac{1}{2}x^2+\frac{x^3}{18}+\frac{x^4}{216}+\frac{x^5}{3240}$, &c. inſtead of z reſtore $y^{\frac{1}{3}}$, and you will have the deſired relation between x and y. That is $y^{\frac{1}{3}}=\frac{1}{2}x^2+\frac{1}{18}x^3+\frac{1}{216}x^4$, &c. and by cubing each ſide $y=\frac{1}{8}x^6+\frac{1}{24}x^7+\frac{1}{288}x^8$, &c.

In like manner if the equation $\dot{y}=\sqrt{4y}+\sqrt{xy}$ were given, or $\dot{y}=2y^{\frac{1}{2}}+x^{\frac{1}{2}}y^{\frac{1}{2}}$; I make $z=y^{\frac{1}{2}}$, or $zz=y$, and thence by PROB. I. $2\dot{z}z=\dot{y}$, and by conſequence $2\dot{z}z=2z+x^{\frac{1}{2}}z$, or $\dot{z}=1+\frac{1}{2}x^{\frac{1}{2}}$. Therefore by the firſt caſe of this, it is $z=x+\frac{1}{3}x^{\frac{3}{2}}$, or

$y^{\frac{1}{2}}=x+\frac{1}{3}x^{\frac{3}{2}}$, then by ſquaring each ſide $y=xx+\frac{2}{3}x^{\frac{5}{2}}+\frac{1}{9}x^{3}$. And if I ſhould deſire to have the value exhibited an infinite number of ways, make $z=c+x+\frac{1}{3}x^{\frac{3}{2}}$, aſſuming any initial term c. and it will be zz, that is, $y=c^{2}+2cx+\frac{2}{3}cx^{\frac{3}{2}}+x^{2}+\frac{2}{3}x^{\frac{5}{2}}+\frac{1}{9}x^{3}$. But perhaps I may ſeem too minute in treating of ſuch things as will but ſeldom come into practice.

Solution of Caſe III.

The reſolution will ſoon be diſpatched, when the equation involves three or more Fluxions of quantities. For between any two of theſe quantities any relation may be aſſumed, when it is not determined by the ſtate of the Queſtion, and the relation of the Fluxions may be found from thence: ſo that either of them together with its Fluxion may be exterminated. For which reaſon, if there be found the Fluxions of three quantities, only one equation need be aſſumed; two, if there be four; and ſo on: that the equation may finally be transformed into another equation, in which two Fluxions only may be found; and then this equation being reſolved as before, the relation of the other quantities may be diſcovered.

Let the equation propoſed be $2\dot{x}-\dot{z}+\dot{y}x=0$; that I may obtain the relation of the quantities x, y, and z, whoſe Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$, are contained in the equation; I form a relation at pleaſure between any two of them, as x and y, ſuppoſing that $x=y$, or $2y=a+z$, or $x=yy$, &c. as ſuppoſe at preſent $x=yy$, and thence $\dot{x}=2\dot{y}y$. Therefore writing $2\dot{y}y$ for $\dot{x}$, and yy for x, the equation propoſed will be transformed into this $4\dot{y}y-\dot{z}+\dot{y}y^{2}=0$; hence

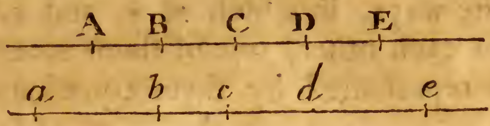
hence the relation between y and z will arise $2yy + \frac{1}{3}y^3 = z$, in which if x be written for yy, and $x^{\frac{3}{2}}$ for y^3, we shall have $2x + \frac{1}{3}x^{\frac{3}{2}} = z$. So that among the infinite ways, in which x, y, and z, may be related to each other, one of them is here found, which is represented by these equations, $x = yy$, $2y^2 + \frac{1}{3}y^3 = z$, and $2x + \frac{1}{3}x^{\frac{3}{2}} = z$.

Demonstration.

And thus we have resolved the Problem, but the Demonstration is still behind. And in so great a variety of matters, that we may not derive it synthetically, and with too great perplexity, from its genuine foundation; it may be sufficient to point it out short by way of analysis, that is, when any equation is proposed after you have finished the work, you may try whether from the derived equation you can turn back to the equation proposed by PROB. I; and therefore the relation of the quantities in the derived equation requires the relation of the Fluxions in the proposed equation; and contrariwise, *Q. E. D.*

So if the equation proposed were $\dot{y} = x$, the derived equation will be $y = \frac{1}{2}x^2$; and on the contrary by PROB. I. we have $\dot{y} = \dot{x}x$, that is $\dot{y} = x$, because $\dot{x}$ is supposed $= 1$. And thus from $\dot{y} = 1 - 3x + y + xx + xy$, is derived $y = x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$, &c. And thence again by PR. I. $\dot{y} = 1 - 2x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{2}{15}x^5$, &c. which two values of $\dot{y}$ agree with each other, as appears by substituting $x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5$, &c. instead of y in the first value.

But in the reduction of equations, I made use of an operation, of which also it will be proper to give ſome account, and that is the tranſmutation of a flowing quantity by its connexion with a given quantity. Let AE and *ae* be two lines indefi-



nitely extended each way, along which two moving things or points paſſing from afar, at the ſame time touch the places A and *a*, B and *b*, C and *c*, D and *d*, *&c.* and let B be the point, by its diſtance from which, the motion of the moving thing or point in AE is eſtimated; ſo that —BA, BC, BD, BE, ſucceſſively may be the flowing quantities, when the thing moving is in the places A, C, D, E. Likewiſe let *b* be a like point in the other line. Then will —BA and —*ba* be contemporaneous Fluents, as alſo BC and *bc*, BD and *bd*, BE and *be*, *&c.* Now if, inſtead of the points B and *b*, be ſubſtituted A and *c*, to which as at reſt the motions are referred; then o and —*ca*, AB and —*cb*, AC and o, AD and *cd*, AE and *ce*, will be contemporaneous flowing quantities. Therefore the flowing quantities are changed by the addition and ſubtraction of the given quantities AB and *ac*: But they are not changed as to the celerity of their motion and the mutual reſpects of their Fluxions; for the contemporaneous parts AB and *ab*, BC and *bc*, CD and *cd*, DE and *de*, are of the ſame length in both caſes. And in equations in which theſe quantities are repreſented, the contemporaneous parts of quantities are not therefore changed, notwithſtanding their abſolute magnitude may be increaſed or diminiſhed by ſome given quantity. Hence the thing propoſed is manifeſt: for the only ſcope of this Problem

blem is to determine the contemporaneous parts or the contemporaneous differences of the abſolute quantities v, x, y, or z, deſcribed by a given rate of flowing; and it is all one of what abſolute magnitude thoſe quantities are, ſo their contemporaneous or correſpondent differences may agree with the propoſed relation of the Fluxions.

The reaſon of this matter may alſo be thus explained algebraically. Let the equation $\dot{y}=x\dot{x}y$ be propoſed, and ſuppoſe $x=1+z$. Then by PR. I. $\dot{x}=\dot{z}$, ſo that for $\dot{y}=x\dot{x}y$ may be wrote $\dot{y}=\dot{x}y+\dot{x}zy$. Now ſince $\dot{x}=\dot{z}$, it is plain that though the quantities, x, and z, be not of the ſame length, yet that they flow alike in reſpect of y, and that they have equal contemporaneous parts; why may I not therefore repreſent by the ſame ſymbols, quantities that agree in their rate of flowing? and to determine their contemporaneous differences, why may not I uſe $\dot{y}=\dot{x}y+\dot{x}zy$ inſtead of $\dot{y}=x\dot{x}y$?

Laſtly, It appears plainly in what manner the contemporary parts may be found from an equation involving flowing quantities: thus if $y=\frac{1}{x}+x$ be the equation; when $x=2$, then $y=2\frac{1}{2}$, but when $x=3$, then $y=3\frac{1}{3}$; therefore while x flows from 2 to 3, y will flow from $2\frac{1}{2}$ to $3\frac{1}{2}$; ſo that the parts deſcribed in this time are $3-2=0$, and $3\frac{1}{3}-2\frac{1}{2}=\frac{5}{6}$. ——This foundation being thus laid for what follows, I ſhall now proceed to more particular Problems.

PROBLEM III.

To determine the Maxima and Minima of Quantities.

When a quantity is the greateſt or the leaſt that it can, be at that moment it neither flows backwards nor forwards : for if it flows forwards or increaſes then it was leſs, and will preſently be greater than it is ; and on the contrary if it flows backwards or decreaſes, then it was greater, and will preſently be leſs than it is. Wherefore find its Fluxion by PROB. I. and ſuppoſe it to be equal to nothing.

EXAMPLE. 1. If in the equation $x^3 - ax^2 + axy - y^3 = 0$ the greateſt value of x be required, find the relation of the Fluxions of x and y, and you will have $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$, then make $\dot{x} = 0$ there will remain $3\dot{y}y^2 + a\dot{y}x = 0$, or $3y^2 = ax$, by the help of which you may exterminate either x or y out of the primary equation ; and by the reſulting equation you may determine the other, and then both of them by $+3y^2 + ax = 0$. This operation is the ſame as if I had multiplied the terms of the propoſed equation by the number of dimenſions of the other flowing quantity y, from whence we may derive that famous Rule of *Huddenius*, *viz.* that in order to obtain the greateſt or leaſt Relate Quantity, the equation muſt be diſpoſed according to the dimenſions of the Correlate Quantity, and then the terms are to be multiplied into any arithmetical progreſſion : but ſince neither this rule, nor any other that I know yet publiſhed extends to equations affected with ſurd quantities without a previous reduction, I will give the following example for that purpoſe.

Ex. 2.

Ex. 2. If the greateſt value of y in the equation $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay+xx} = 0$ be to be determined, ſeek the Fluxions of x and y, and there will ariſe the equation $3\dot{x}x^2 - 2a\dot{y}y + \frac{3ab\dot{y}y^2 + 2b\dot{y}y^3}{a^2+2ay+y^2} - \frac{4a\dot{x}xy - 6\dot{x}x^3 - a\dot{y}x^2}{2\sqrt{ay+xx}} = 0$; and ſince by ſuppoſition $\dot{y}=0$, omit the terms multiplied by $\dot{y}$, (which, to ſhorten the labour, might have been done in the operation,) and divide the reſt by $\dot{x}x$, then there will remain $3x - \frac{2ay+3xx}{\sqrt{ay+xx}} = 0$, and when the reduction is made there will ariſe $4ay + 3xx = 0$, by the help of which you may exterminate either of the quantities x or y out of the propoſed equation; and then from the reſulting equation, which will be cubical, you may extract the value of the other.

From this Problem may be had the ſolution of theſe following.

1. *In a given triangle, or in the ſegment of any given curve, to inſcribe the greateſt rectangle.*

2. *To draw the greateſt or the leaſt right line, which can lye between a given point and a curve given in poſition, or to draw a perpendicular to a curve from a given point.*

3. *To draw the greateſt or the leaſt right line which paſſing through a given point can lye between two others, either right lines or curves.*

4. *From a given point within a parabola, to draw a right line which ſhall cut the parabola more obliquely than any other, and to do the ſame in other curves.*

5. *To determine the vertices of curves, their greateſt or leaſt breadths, the points in which revolving parts cut each other, &c.*

6. *To*

6. *To find the points in curves where they have the greateſt or leaſt curvature.*

7. *To find the leaſt angle in a given ellipſis, in which the ordinates can cut their diameters.*

8. *Of ellipſes that paſs through four given points, to determine the greateſt, or that which approaches neareſt to a circle.*

9. *To determine the amplitude of a ſpherical ſuperficies, which can be illuminated in its poſterior part, by light coming from a great diſtance, and which is refracted by the anterior hemiſphere.*

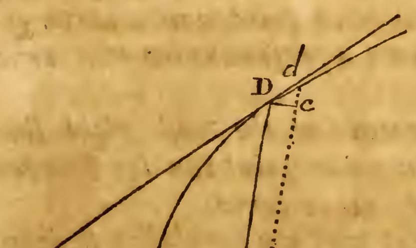
And many other Problems of like nature may more eaſily be propoſed than reſolved, becauſe of the labour of the Computation.

PROBLEM IV.

To draw Tangents to Curves.

The Firſt manner.

Tangents may be variouſly drawn according to the various relation of curves to right lines: and firſt, let BD be a right line or ordinate in a given angle to another right line AB, as a baſe or abſciſs, and terminated at the curve ED; let this ordinate move thro' an indefinite ſmall ſpace to the place *bd*, ſo that it may be increaſed by the moment *cd*, while AB is increaſed by the moment B*b* to which DC is equal and parallel, let D*d* be produced till it meet with AB in T, and this line will touch the curve in D or *d*, and the triangles *dc*D, DBT will



will be similar; so that TB . BD : : D*c*, or B*b* : *cd*. Since therefore the relation of BD to AB is exhibited by the equation by which the nature of the curve is determined, seek for the relation of the Fluxion by PROB. I. Then take TB to BD in the ratio of the Fluxion of AB to the Fluxion of BD, and TD will touch the curve in the point D.

EXAMPLE 1. Calling AB$=x$ and BD$=y$, let their relations be $x^3-ax^2+axy-y^3=0$, and the relation of the Fluxion will be $3\dot{x}x^2-2a\dot{x}x+a\dot{x}y-3\dot{y}y^2+a\dot{y}x=0$, so that $\dot{y}:\dot{x}::3xx-2ax+axy+ay:3y^2-ax$: : BD or (y) : BT. Therefore BT $=\frac{3y^3-axy}{3x^2-2ax+ay}$; therefore the point D being given, and thence DB and AB, or y and x, the length will be given by which the tangent TD is determined.

But this method of operation may be thus concinnated: make the terms of the proposed equation equal to nothing, then multiply by the proper number of the dimension of the ordinate, and put the result in the numerator; then multiply the same equation by the proper number of the dimensions of the abscis, and put the product divided by the abscis in the denominator of the value of BT; then take BT towards A if this value be affirmative, but the contrary way if the value be negative.

Thus the equation $\overset{0}{\underset{3}{x^3}}-\overset{0}{\underset{2}{ax^2}}+\overset{1}{\underset{1}{axy}}-\overset{3}{\underset{0}{y^3}}=0$, being multiplied by the upper numbers gives $axy-3y^3$ for the numerator, and multiplied by the lower numbers, and then divided by x, gives $3x^2-2ax+ay$ for the denominator of the value of BT.

Thus the equation $y^3-by^2-cdy+bcd+dxy=0$ (which denotes a parabola of the second kind, by help

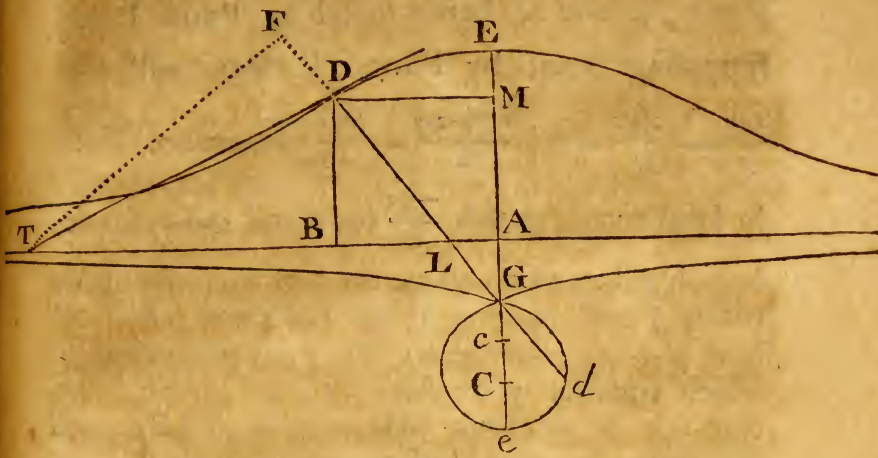
help of which *Des Cartes* constructed equations of six dimensions. See his *Geom. pag.* 42.) by inspection gives $\frac{3y^3-2by^2-cdy+dxy}{dy}$ or $\frac{3yy}{d}-\frac{2by}{d}-c+x$ $=BT$. And thus the equation $a^2-\frac{r}{q}x^2-y^2=0$ (which denotes an ellipsis whose center is A) gives $\frac{-2yy}{-\frac{2r}{q}x}$ or $\frac{qyy}{rx}=BT$. And so in others.

And you may take notice, that it matters not of what quantity the angle of ordination ABD may be. But as this rule does not extend to equations affected by surd quantities or to mechanical curves, in this case we must have recourse to the fundamental method.

Ex. 2. Let $x^3-ay^2+\frac{by^3}{a+y}-xx\sqrt{ay+xx}=0$ be the equation expressing the relation between AB and BD, and by PROB. I. the relation of the Fluxions will be $3\dot{x}x^2-2a\dot{y}y+\frac{3ab\dot{y}y^2+2b\dot{y}y^3}{aa+2ay+yy}-\frac{4a\dot{x}xy-6\dot{x}x^3-a\dot{y}x^2}{2\sqrt{ay+xx}}=0$, therefore it will be $3xx-\frac{4axy+6x^3}{2\sqrt{ay+xx}}:2ay-\frac{3abyy+2by^3}{aa+2ay+yy}+\frac{axx}{2\sqrt{ay+xx}}$ $::\dot{y}:\dot{x}::BD:BT$.

Ex. 3. Let ED be the Conchoid of *Nicomedes* described with the pole G, the Asymptote AT, and the distance LD; and let $GA=b$, $LD=c$, $AB=x$, and $BD=y$. Then because of the similar triangles DBL and DMG, it will be LB . BD : : DM : MG : that is, $\sqrt{cc-yy}:y::x:b+y$, and therefore $\overline{b+y}\sqrt{cc-yy}=yx$. Having this equation, I suppose $\sqrt{cc-yy}=z$, and thus I shall have two equations $bz+yz=yx$, and $zz=cc-yy$, by the help

help of theſe I find the Fluxions of the quantities x, y, and z, by PROB. I. from the firſt ariſes $b\dot{z}+y\dot{z}$



$+\dot{y}z=\dot{y}x+\dot{x}y$, and from the ſecond $2\dot{z}z=-2\dot{y}y$ or $\dot{z}z+\dot{y}y=0$, out of theſe if we exterminate $\dot{z}$ there will ariſe $\frac{-b\dot{y}y}{z}-\frac{\dot{y}y^2}{z}+\dot{y}z=\dot{y}x+\dot{x}y$, which being reſolved it will be $y : z-\frac{by}{z}-\frac{yy}{z}-x :: (\dot{y} : \dot{x} ::)$ BD : BT. But as BD is y, therefore BT $=z-x-\frac{by-yy}{z}$, that is, $-\text{BT}=\text{AL}+\frac{\text{BD}\times\text{GM}}{\text{BL}}$ where the ſign — being prefixed to BT denotes that the point T muſt be taken the contrary way to the point A.

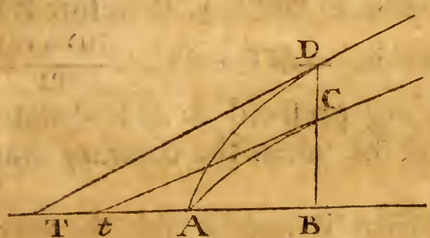
Scholium. And hence it appears by the way, how that point of the Conchoid muſt be found which ſegregates the concave from the convex part; for when AT is the leaſt poſſible D will be the point. Therefore make AT$=v$, and ſince BT$=$ $-z+x+\frac{by+yy}{z}$, then $v=-z+2x+\frac{by+yy}{z}$, here to ſhorten the work, for x I ſubſtitute $\frac{bz+yz}{y}$

 (which

(which value is derived from what goes before,) and it will be $\frac{2bz}{y}+z+\frac{by+yy}{z}=v$, whence the Fluxions $\dot{v}$, $\dot{y}$, and $\dot{z}$, being found by PROB. I. and supposing $\dot{v}=0$, by PROB. III. there will arise $\frac{2b\dot{z}}{y}-\frac{2b\dot{y}z}{yy}+\dot{z}+\frac{b\dot{y}+2\dot{y}y}{z}-\frac{b\dot{z}y-\dot{z}yy}{zz}=\dot{v}=0$. Lastly substituting in this $\frac{-\dot{y}y}{z}$ for $\dot{z}$ and $cc-yy$ for zz, (which values of $\dot{z}$ and zz are had from what goes before,) and making a due reduction, we shall have $y^3+3by^2-2bc^2=0$; by the construction of which equation y or AM will be given: then through M drawing MD parallel to AB it will fall upon the point D of contrary flexure.

Now if the curve be mechanical whose tangent is to be drawn, the Fluxions of the quantities are to be found as in Ex. 5. PROB. I. and then the rest is to be perform'd as before.

Ex. 4. Let AC and AD be two curves which are cut in the points C and D by the right line BCD applied to the abscis AB in a given angle; let $AB=x$, $BD=y$, and area $ACB=z$, then by PROB. I. Preparation to



Ex. 5. it will be $\dot{z}=\dot{x}\times BC$.

Now let AC be a circle or any known curve, and to determine the other curve AD let an equation be proposed, in which z is involved $zz+axz=y^4$; then by PROB. I. $2\dot{z}z+a\dot{x}z+ax\dot{z}=4\dot{y}y^3$, and writing $\dot{x}\times BC$ for $\dot{z}$ it will be $2\dot{x}z\times$

BC

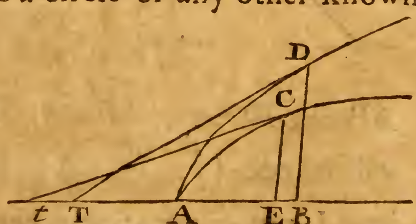
$BC + a\dot{x}x \times BC + a\dot{x}z - 4\dot{y}y^3 = 0$, therefore $2z \times BC + ax \times BC + az : 4y^3 :: (\dot{y} : \dot{x} ::)$ BD : BT; so that if the nature of the curve AC be given the ordinate BC and the area ACB or z, the point D will be given, through which the Tangent DT will pass.

After the same manner if $3z = 2y$ be the equa- of the curve AD, it will be $3\dot{z}$ (or $3\dot{x} \times BC$) $= 2\dot{y}$, so that $3BC : 2 :: (\dot{y} : \dot{x} ::)$ BD : BT. And so in others.

Now for determining the other curve AD whose tangent is to be drawn, let there be given an equation in which z is involved, suppose $z = y$, then it will be $\dot{z} = \dot{y}$, and $Ct : Bt :: (\dot{y} : \dot{x} ::)$ BD : BT, but the point T being found, the Tangent DT may be drawn.

Ex. 5. Let $AB = x$, $BD = y$, as before, and let the length of any curve AC be z, and drawing a tangent to it, as Ct, it will be $Bt : Ct :: \dot{x} : \dot{z}$ or $\dot{z} = \frac{\dot{x} \times Ct}{Bt}$: As suppose $xz = yy$, it will be $\dot{x}z + \dot{z}x = 2\dot{y}y$, and for $\dot{z}$ writing $\frac{\dot{x} \times Ct}{Bt}$, there will arise $\dot{x}z + \frac{\dot{x}x \times Ct}{Bt} = 2\dot{y}y$, therefore $z + \frac{x \times Ct}{Bt} :: 2y :$ BD : BT.

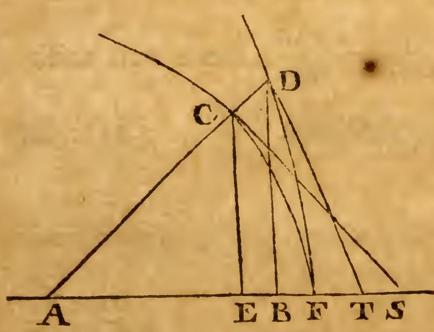
Ex. 6. Let AC be a circle or any other known curve whose tangent is Ct, and let AD be any other curve whose Tangent DT is to be drawn, and let that be defined by assuming AB equal to the arch AC; and CE,



BD being ordinates to AB in a given angle, let the relation of BD to CE or AE be expreffed by any equation. So call AB or AC$=x$, BD$=y$, AE$=z$, and CE$=v$, and it is plain that $\dot{v}$, $\dot{x}$, and $\dot{z}$, the Fluxions of CE, AC, AE, are to each other, as CE, Ct and Et; therefore $\dot{x}\times\frac{CE}{Ct}=\dot{v}$, and $\dot{x}\frac{Et}{Ct}\times=\dot{z}$.

Now let any equation be given to define the curve AD, as $y=z$, then $\dot{y}=\dot{z}$, and therefore Et : Ct : : ($\dot{y}$. $\dot{x}$: :) BD : BT. Or let the equation be $y=z+v-x$, and it will be $\dot{y}=\dot{v}+\dot{z}-\dot{x}=\frac{\dot{x}\times\overline{CE+Et-Ct}}{Ct}$ and therefore CE$+$E$t$$-Ct$: Ct : : ($\dot{y}$: $\dot{x}$: :) BD : BT. Or finally let the equation be $ayy=v^3$, and it will be $2a\dot{y}y=(3\dot{v}v^2=)\ 3\dot{x}v^2\times\frac{CE}{Ct}$, fo that $3v^2\times$ CE : $2ay\times$ Ct : : BD : BT.

Ex. 7. Let FC be a circle which is touched by CS in C, and let FD be a curve which is defined by affuming any relation of the ordinate DB to the arch FC, which is intercepted by DA drawn to the center. Then letting fall CE, the ordinate in the circle, call AC or AF$=1$,

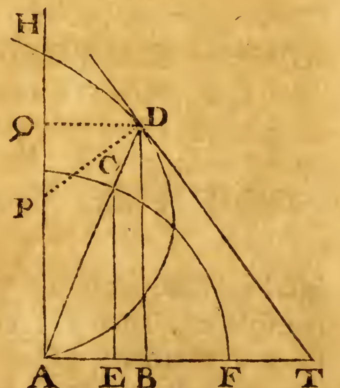


AB$=x$, BD$=y$, AE$=z$, CE$=v$, CF$=t$, and it will be $\dot{t}z=(\dot{t}\times\frac{CE}{CS}=)\ \dot{v}$, and $-\dot{t}v=(\dot{t}\times\frac{ES}{CS}=)\dot{z}$, here I put $\dot{z}$ negatively becaufe AE is diminifhed while EC is increafed; and befides AE : EC : :

AB

AB : BD, ſo that $zy = vx$, and thence by PROB. I $\dot{z}y + \dot{y}z = \dot{v}x + \dot{x}v$, then exterminating $\dot{v}$, $\dot{z}$, and v, it is $\dot{y}x - \dot{t}y^2 - \dot{t}x^2 = \dot{x}y$. Now let the curve DF be defined by an equation, from which the value of $\dot{t}$ may be derived to be ſubſtituted here: ſuppoſe let $t = y$, (an equation to the firſt quadratrix,) and by PROB. I. it will be $\dot{t} = \dot{y}$, ſo that $\dot{y}x - \dot{y}y^2 - \dot{y}x^2 = \dot{x}y$, whence $y \,.\, xx + yy - x :: (\dot{y} : -\dot{x} ::)$ BD (y) : BT, therefore BT $= x^2 + y^2 - x$, and AT $= xx + yy = \frac{ADq}{AF}$. ——After the ſame manner if it is $tt = by$, there will ariſe $2\dot{t}t = b\dot{y}$, and thence AT $= \frac{b}{2t} \times \frac{ADq}{AF}$. And ſo of others.

Ex. 8. Now if AD be taken equal to the arch FC, (the curve ADH being then the ſpiral of *Archimedes*) the ſame names of the lines ſtill remaining as were put afore; becauſe of the right angle ABD it is $xx + yy = tt$, and therefore by PROB. I. $\dot{x}x + \dot{y}y = \dot{t}t$; it is alſo AD : AC : : DB : CE, ſo that $tv = y$, and thence by PROB. I. $\dot{t}v + t\dot{v} = \dot{y}$: laſtly the Fluxion of the arch FC is to the Fluxion of the right line CE : : AC : AE, or as AD : AB, that is, $\dot{t} : \dot{v} : t : x$, and thence $\dot{t}x = \dot{v}t$; compare the equations now found, and you will have $\dot{t}v + \dot{t}x = \dot{y}$, and thence $\dot{x}x + \dot{y}y = (\dot{t}t =) \frac{\dot{y}t}{v + x}$. Therefore com-



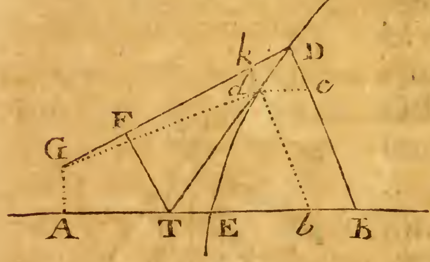
pleating

pleating the parallelogram ABDQ, if you make QD : QP : : (BD : BT : : $y : x$: :) $x : y - \frac{t}{b+x}$ that is, if you take AP$=\frac{t}{v+x}$, PD will be perpendicular to the ſpiral.

And from hence I imagine it will be ſufficiently manifeſt by what method the tangents of all ſorts of curves may be drawn; however it may not be foreign to the purpoſe, if I alſo ſhow how the Problem may be perform'd when the curves are referred to right lines after any other manner whatever, ſo that here being the choice of ſeveral methods, the eaſieſt and moſt ſimple may always be uſed.

The Second manner.

Let D be a point in the curve, from whence the ſubtenſe DG is drawn to a given point G, and let DB be an ordinate in any given angle to the abſciſs AB. Now let the point D flow thro' an infinitely ſmall ſpace D*d* in the curve, and in GD let G*k* be taken equal to G*d*, and let the parallelogram *dc*B*b*, be completed, then D*k* and D*c* will be the contemporary moments of GD and BD by which they are diminiſhed, while D is transferred to *d*. Now let the right line FT be produced, till it meet with AB in T, and from the point T to the ſubtenſe GD, let fall the perpendicular TF, and then the trapezia D*cdk* and DBTF will be alike; and therefore DB : DF : : D*c* : D*k*. Since then the relation of BD to GD is exhibited by the equation for determining the curve, find the relation of the Fluxion,



ions, and take FD to DB in the ratio of the Fluxion of GD to the Fluxion of BD, then from F raise the perpendicular FT which may meet with AB in T, and drawing DT it will touch the curve in D, but DT must be taken towards G, if it be affirmative, and the contrary way, if it be negative.

Ex. 1. Call GD$=x$, and BD$=y$, and let their relation be $x^3-ax^2+axy-y^3=0$; then the relation of the Fluxions will be $3\dot{x}x^2-2a\dot{x}x+a\dot{x}y+a\dot{y}x-3\dot{y}y^2=0$. Therefore $3xx-2ax+ay : 3yy-ax :: (\dot{y} : \dot{x} ::)$ DB (y) DF, so that DF $=\frac{3y^3-axy}{3x^2-2ax+ay}$. Then any point D in the curve being given, and thence BD and GD or y and x, the point will be given also, from whence if the perpendicular FT be raised, from its concourse with the abscis AB in T, the Tangent DT must be drawn.

And hence it appears, that a rule may be derived here, as well as in the former case. For having disposed all the terms of the given equation on one side, multiply its terms severally by the dimensions of the ordinate y, and place the result in the numerator of the fraction; then multiply by the dimensions of the subtense x, and dividing the result by that subtense x, place the quote in the denominator of the value of DF; and take the same line DF towards G, if it be affirmative, otherwise the contrary way. Where you may observe that it is no matter, how far distant the point G is from the abscis AB; or if it be at all distant: or what is the angle of ordination ABD.

Let the equation be as before $x^3-ax^2+axy-y^3=0$; it gives immediately $axy-3y^3$ for the numerator, and $3x^2-2ax+ay$ for the denominator of the value of DF.

Let

Let alſo $a + \frac{b}{a}x - y = 0$, (which equation is to a Conick Section,) it gives $-y$ for the numerator, and $\frac{b}{a}$ for the denominator of the value of DF, which therefore will be $-\frac{ay}{b}$.

And thus in the Conchoid, (in which the matter will be performed more expeditiouſly than before) putting $GA = b$, $LD = c$, $GD = x$, and $BD = y$, [*See Fig. pag.* 65.] it will be BD (y) : DL (c) : : GA (b) : GL $(x - c)$, therefore $xy - cy = cb$, or $xy - cy - cb = 0$. This equation according to the rule gives $\frac{xy - cy}{y}$, ſo that $x - c = DF$. Therefore prolong GD to F, ſo that DF=LG, and at F raiſe the perpendicular FT, meeting the aſymptote AB in T, then DT being drawn which touch the Conhoid.

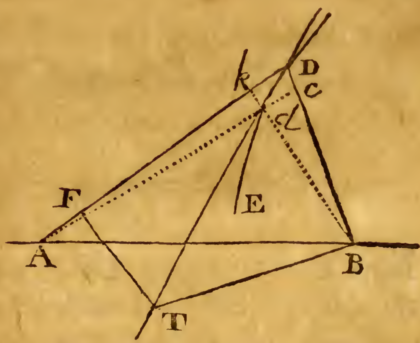
But when compound or ſurd quantities are found in the equation, you muſt have recourſe to the general method, except you ſhould chooſe rather to reduce the equation.

Ex. 2. If the equation $b + y\sqrt{cc - yy} = yx$, were given for the relation between GD and BD; [*See Fig. pag.* 70.] find the relation of the Fluxions by PROB. I. as ſuppoſe $\sqrt{cc - yy} = z$, and you will have the equations $bz + yz = yx$ and $cc - yy = zz$, and thence the relation of the Fluxions $b\dot{z} + \dot{y}z + y\dot{z} = \dot{y}x + y\dot{x}$, and $-2\dot{y}y = 2\dot{z}z$. Now z and $\dot{z}$ being exterminated, there will ariſe $\dot{y}\sqrt{cc - yy} - \frac{b\dot{y}y - \dot{y}y^2}{\sqrt{cc - yy}} = \dot{y}x + \dot{x}y$; therefore $y : \sqrt{cc - yy} - \frac{by - yy}{\sqrt{cc - yy}} - x : : (\dot{y} : \dot{x} : :)$ BD (y) : DF.

The

Third manner.

Moreover if the Curve be referred to two ſubtenſes AD and BD, which being drawn from two given points A and B, may meet at the Curve: Conceive that point D to flow on thro' an infinitely little ſpace D*d* in the curve, and in AD and BD take A*k*=A*d* and B*c*=B*d*, and then *k*D and *c*D will be contemporaneous moments of the lines AD and BD. Take therefore DF to BD in the ratio of the moment D*k* to the moment D*c*, (that is in the ratio of the Fluxion of the line AD to the Fluxion of the line BD,) and erect the perpendiculars BT FT meeting in T, then the trapezia DFTB and D*kdc* will be ſimilar, and therefore the diagonal DT will touch the Curve.

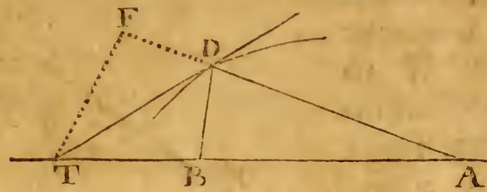


Therefore from the equation by which the relation is defined between AD and BD, find the relation of the Fluxions by PROB. I. and take FD to BD in the ſame ratio.

Ex. Suppoſing AD=x and BD=y, let their relation be $a+\frac{ex}{d}-y=0$; (this equation is to the ellipſes of the ſecond order, whoſe properties for refracting light are ſhewn by *Des Cartes* in the ſecond book of his Geometry.) Then the relation of the Fluxions will be $\frac{e\dot{x}}{d}-\dot{y}=0$. Thus therefore $e.d::\dot{y}:\dot{x}::$ BD : DF; and for the ſame reaſon if $a-\frac{ex}{d}-y=0$, it will be $e:-d::$ BD : DF.

DF. In the firſt caſe take DF towards A, and the contrary way in the other caſe.

Corol. 1. Hence if $d=e$, (in which caſe the curve becomes a Conick Section,) it will be DF=DB.



And the triangles DFT and DBT being equal, the angle FDB will be biſected by the tangent.

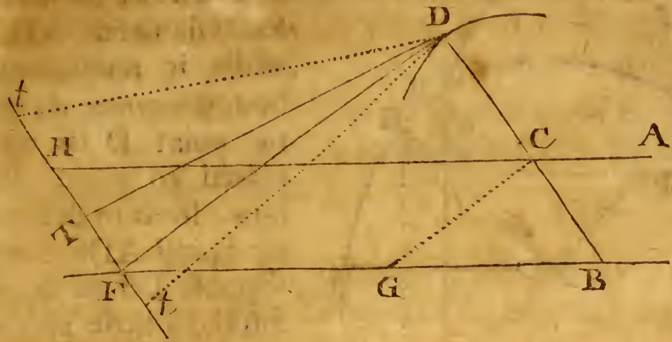
Corol. 2. Hence thoſe things will be manifeſt of themſelves, which are demonſtrated by *Des Cartes*, concerning the Refraction of this curve, in a very prolix manner. Foraſmuch as DF and DB, (which are in the given ratio of d to e) in reſpect of the radius DT, are the ſines of the angles DTF and DTB, that is of the ray of incidence AD upon the ſurface of the Curve, and of its Reflexion or Refraction DB. And there is a like reaſoning concerning the Refractions of the Conic Sections, ſuppoſe that either of the points, A, or B, be conceived to be at an infinite diſtance.

It would be eaſy to modify this rule in the manner of the foregoing, and to give more examples of it; as alſo when curves are referred after any other manner, and cannot commodiouſly be reduced to the foregoing, it will be very eaſy to find out other methods in imitation of this, as occaſion ſhall require.

The Fourth manner.

As if the right line BCD ſhould revolve about a given point B, and one of its points D ſhould deſcribe

ſcribe a Curve, and another point C ſhould be the interſection of the right line BCD with another



right line AC given in poſition. Then the relation of BC and BD being expreſſed by any equation; draw BF parallel to AC, ſo as to meet DF perpendicular to BD in F. Alſo erect FT perpendicular to DF, and take it in the ſame ratio to BC, as the Fluxion of BD has to the Fluxion of BC. Then drawing DT it will touch the Curve.

The Fifth manner.

But if the point A being given, the equation ſhould expreſs the relation between AC and BD; draw CG parallel to DF, and take FT in the ſame ratio to BG, as the Fluxion of BD has to the Fluxion of AC.

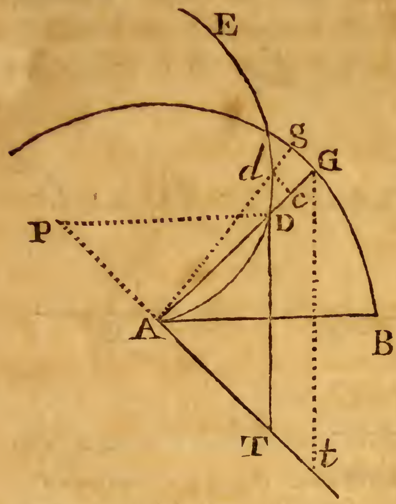
The Sixth manner.

Or again, if the equation expreſſes the relation between AC and CD; let AC and FT meet in H, and take HT in the ſame ratio to BG, as the Fluxion of CD has to the Fluxion of AC. And the like in others.

The Seventh manner. *For* SPIRALS.

The Problem is not different when the Curves are referred not to Right Lines, but to other Curve

Lines, as is usual in Mechanical Curves. Let BG be the Circumference of a Circle, in whose semi-diameter A G, while it revolves about the center A, let the point D be conceived to move any how, so as to describe the Spiral ADE; and suppose D*d* to be an infinitely little part of the Curve thro' which D flows; then in AD take A*c* equal to A*d*, and *c*D and G*g* will be the contemporary moments of the right line AD and the periphery BG. Therefore draw A*t* parallel to *cd*, that is, perpendicular to AD, and let the tangent DT meet it in T. Then it will be CD : *cd* : : AD : AT. Also let G*t* be parallel to the tangent DT, and it will be *cd* : G*g* : : A*d*, or AD : AG : : AT : A*t*.



Therefore any equation being proposed, by which the relation is expressed between BG and AD; find the relation of their Fluxions by PROB. I. and take A*t* in the same ratio to AD; then G*t* will be parallel to the Tangent.

Ex. 1. Calling BG$=x$ and AD$=y$, let their relations be $x^3-ax^2+axy-y^3=0$, and by PROB. I. $3x^2-2ax+ay : 3y^2-ax : : (\dot{y} : \dot{x} : :)$ AD : A*t* : : AP : AG. The point *t* being thus found, draw G*t*, and DT parallel to it, which will touch the Curve.

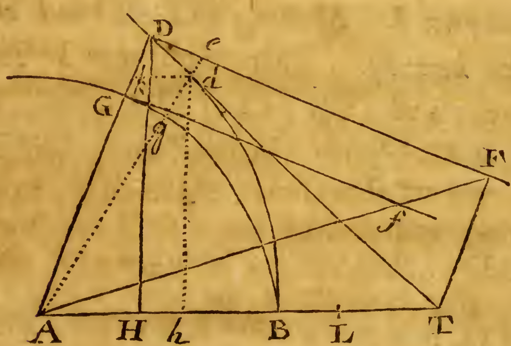
Ex.

Ex. 2. If it is $\frac{ax}{b}=y$; (which is the equation to the Spiral of *Archimedes)* it will be $\frac{a\dot{x}}{b}=\dot{y}$. Therefore $a:b::(\dot{y}:\dot{x}::)$ AD : At. Wherefore, by the way, if TA be produced to P, that it may be AP : AD : : $a:b$; then PD is Perpendicular to the Curve.

Ex. 3. If $xx=by$, then $2\dot{x}x=b\dot{y}$, and $2x:b::$ AD : At. And thus Tangents may be easily drawn to any Spirals whatsoever.

The Eighth manner. *For* QUADRATRICES.

Now if a Curve be such, that any line AGD, being drawn from the center A, meets the Circle



in G, and the Curve in D: and if the relation between the arch BG and the right line DH, which is an ordinate to the base or abscis AH in a given angle, be determined by any equation whatever: Conceive the point D to move in the Curve for an infinitely little interval to d, and the parallelogram dbHk being compleated, produce Ad to e, so that Ae=AD. Then Gg and Dk will be contemporaneous

temporaneous moments of the arch BG and the ordinate DH. Now produce D*d* to T, where it may meet with AB, and from thence let fall the perpendicular TF on D*e*F. Then the trapezia D*kde* and DHTF will be similar; therefore D*k* : *de* : : DH : DF. And befides if G*f* be raifed perpendicular to AG meeting AF in *f*, becaufe of the parallels DF and G*f*, it will be D*e* : G*g* : : DF : G*f*; therefore *ex æquo* D*k* : G*g* : : DH : G*f*, that is as the moments or Fluxions of the lines DH and B G. Therefore by the equation which expreffes the relation of BG to DH, find the relation of the Fluxions by PROB. I. and in that ratio take G*f* (the tangent of the circle BG) to DH; draw DF parallel to G*f*, which may meet A*f* produced in F, and at F erect the perpendicular FT meeting AB in T, then the right line DT being drawn will touch the Quadratrix.

EXAMPLE I. Making BG$=x$, and DH$=y$; let it be $xx=by$. Then by PROB. I. $2x\dot{x}=b\dot{y}$; therefore $2x : b : : (\dot{y} : \dot{x} : :)$ DH : G*f*, but the point F being found, the reft will be determined as above. ——But perhaps this rule may be made fomething neater. Make $\dot{x} : \dot{y}$: : AB : AL. Then AL : AD : : AD : AT; and DT will touch the curve For becaufe of the equal triangles AFD and ATD, it is AD$\times$DF$=$AT$\times$DH, and therefore AT : AD : : (DF : DH or $\frac{\dot{y}}{\dot{x}}$ G*f* : :) AD : $\frac{\dot{y}}{\dot{x}}$ AG or AL.

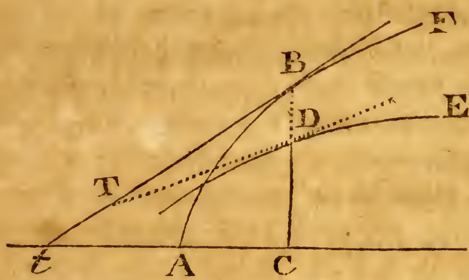
Ex. 2. Let $x=y$; (which is the equation to the Quadratrix of the Antients;) then $\dot{x}=\dot{y}$; therefore AB : AD : : AD : AT.

Ex.

Ex. 3. Let $axx = y^3$. Then $2a\dot{x}x = 3\dot{y}y^2$: therefore make $3y^2 : 2ax :: (\dot{x} : \dot{y} ::)$ AB : AL; then AL : AD : : AD : AT. And thus you may determine expeditiously the Tangents of any Quadratrices whatever.

The Ninth manner.

Lastly. If ABF be any given Curve, which is touched by the right line Bt; and a part BD of



the right line BC (being an ordinate in any given angle to the abscis AC) intercepted between this and another Curve DE, has a relation to the Fluxion of the curve AB, which is expressed by any equation; you may draw a Tangent DT to the other Curve, by taking (in the Tangent of this curve) BT in the same ratio to BD, as the Fluxion of the Curve AB has to the Fluxion of the Right Line BD.

EXAMPLE 1. Calling AB $= x$, and BD $= y$, let it be $ax = yy$: therefore $a\dot{x} = 2\dot{y}y$. Then $a : 2y :: (\dot{y} : \dot{x} ::)$ BD : BT.

Ex. 2. Let $\frac{a}{b} = y$. (The equation to the Trochoid, if ABF be a circle.) Then $\frac{a}{b}\dot{x} = \dot{y}$, and $a : b ::$ BD : BT.

And with the same ease may Tangents be drawn, when the relation of BD to AC or to BC is expres-
sed

ſed by any equation. Or when the Curves are referred to Right Lines; or to any other Curves after any other manner whatſoever.

There are alſo many other Problems, whoſe ſolutions are to be derived from the ſame principles. ———Such as theſe following.

1. *To find a point of a curve, where a Tangent is parallel to the baſe; or to any right line given in poſition; or is perpendicular to it; or inclined to it in any given angle.*

2. *To find the point, where the Tangent is moſt or leaſt inclined to the baſe; or any other right line given in poſition; that is to find the Confine of contrary Flexure.* Of this I have given a ſpecimen in the Conchoid.

3. *From any given point without the Perimeter of a curve to draw a right line, which with the Perimeter ſhall make an angle of contact; or a right angle; or any other given angle: that is, from a given point to draw Tangents or perpendiculars, or light lines, that ſhall have any other inclination to a curve line.*

4. *From any given point within a Parabola to draw a right line, which ſhall make, with the Perimeter, the greateſt or leaſt angle poſſible. And ſo of all curves whatſoever.*

5. *To draw a right line which ſhall touch two curves given in poſition; or the ſame curve in two points when that can be done.*

6. *To draw any curve with given conditions, which ſhall touch another curve given in poſition in a given point.*

7. *To determine the refraction of any ray of light, that falls upon any curve ſuperficies.*

The reſolution of theſe, or of any the like Problems, will not be ſo difficult, abating the tediouſneſs of computation, that there is any occaſion to enlarge upon them here. And I imagine it will be more agreeable to Geometricians barely to have mentioned them.

PROB.

PROBLEM V.

At any given Point of a given Curve, to find the quantity of Curvature.

There are few Problems concerning Curves more elegant than This, or that give a greater inſight into their nature. In order to its reſolution, I muſt premiſe the following general conſiderations.

1. The ſame Circle has every where the ſame Curvature, and in different Circles the Curvature is reciprocally proportional to their diameters: If the diameter of any Circle be as little again as that of another Circle, the Curvature of its Periphery will be as great again, if the diameter be a third of the other, the Curvature will be thrice as much, *&c.*

2. If a Circle touches any Curve on its concave ſide in a given point, and its magnitude be ſuch that no other Tangent Circle can be interſcribed in the Angle of contact nearer that point, that Circle will be of the ſame Curvature as the Curve is of in that point of contact. For that circle which comes between the curve and another Circle at the point of contact, varies leſs from the Curve and makes a nearer approach to its Curvature, than that other Circle does; and therefore that Circle approaches neareſt to its Curvature, between which and the Curve no other Circle can intervene.

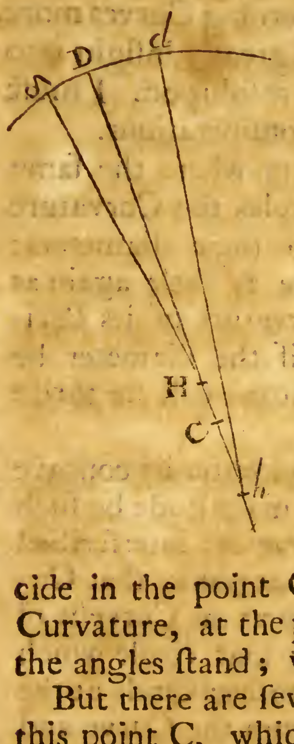
3. Therefore the Center of Curvature at any point of a curve, is the Center of a Circle equally curved, and thus the Radius or Semi-diameter of Curvature is part of the perpendicular which is terminated at that Center.

4. And the proportion of Curvature at different points will be known from the proportion of Cur-

vature of Equi-curved Circles, or from the reciprocal proportion of the Radii of Curvature.

The Problem then is reduced to this, *viz.* To find the Radius or Center of Curvature.

Imagine therefore that at three points of the Curve δ, D, d, perpendiculars are drawn, of which those that are at D, δ, meet in H, and those that are in D, d, meet in h, and the point d being in the middle, if there be a greater curvitude at the part Dδ than at Dd, then DH will be less than dh; but by how much the perpendiculars δH and dh are nearer to the intermediate perpendicular, so much the less will the distance be of the points H and h, and at last, when the perpendiculars meet, the points will coincide. Let them coincide in the point C, and C will be the Center of Curvature, at the point of the Curve D on which the angles stand; which is manifest of itself.



But there are several symptoms or properties of this point C, which may be of use for its determination.

As, 1. That it is the concourse of Perpendiculars, that are on each side at an infinitely little distance from DC.

2. That the intersection of Perpendiculars at any little finite distance on each side, are separated and divided by it; so that those that are on the more curved side Dδ, sooner meet at H, and those that are on the other less curved side Dd, meet more remotely at h.

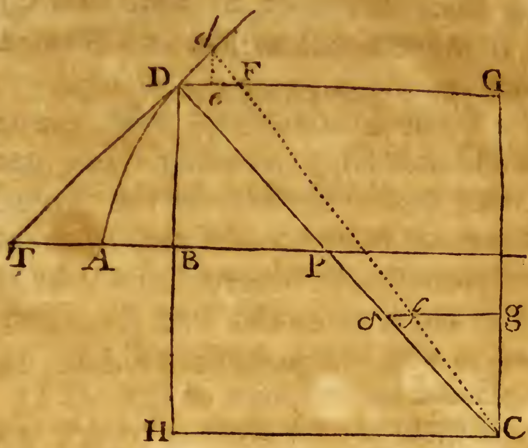
3. If DC be conceived to move, while it insists perpendicularly on the Curve, that point of it C (if you except the motion of its approaching or receding from the point of insistance C) will be least of all moved, but will be as it were the Center of motion.

4. If a Circle be described with the center C and the distance DC, no other circle can be described, that can lie between at the contact.

5. If the center H or *h* of any other touching Circle approaches by degrees to C the center of this, till at last it coincides with it; than any of the points in which that circle will cut the Curve, will coincide with the point of contact D.

Each of these properties may supply means for resolving the Problem different ways; but we shall here make choice of the First, as being the most simple.

At any point D of the Curve AD, let DT be a Tangent, DC a Perpendicular, and C the Center of Curvature, as before. And let AB be the Abscis, to which let DB be applied at right angles, which DC meets in P. Draw DG parallel to AB, and CG perpendicular to it, in which take C*g* of any given magnitude, and draw *g*δ perpendicular to it, which meets DC in δ. Then it will be C*g* . *g*δ :: (TB : BD ::) as the Fluxion of the Abscis to the Fluxion of the Ordinate. Likewise imagine the point D to move in the Curve an infinitely little distance D*d*, and drawing *de* perpendicular to DG, and C*d* perpendicular to the Curve, let C*d* meet DG in F, and δ*g* in *f*. Then will D*e* be the *momentum* of the Abscis, *de* the *momentum* of the Ordinate, and δ*f* the contemporaneous *momentum* of the Right Line *g*δ. Therefore $DF = De + \frac{de \times de}{De}$. Having therefore the ratios of these *momenta*, or which is the same thing, of their generating Fluxions, you will have the ratio of GC to the given line Cg, which is the same as that of



DF to δf. And thence the point C will be determined.

Therefore let AB$=x$, BD$=y$, Cg$=1$, and $g\delta$ $=z$. Then it will be $1 : z :: \dot{x} : \dot{y}$, or $z=\frac{\dot{y}}{\dot{x}}$. Now let the *momentum* δf of z be $\dot{z} \times o$, (that is the product of the velocity and of an infinitely small quantity o,) therefore the *momentum* D$e=\dot{x}\times o$, de $=\dot{y} \times o$, and thence DF$=\dot{x}o+\frac{\dot{y}\dot{y}o}{\dot{x}}$. Therefore it is Cg (1) : CG :: (δf : DF ::) $\dot{z}o : \dot{x}o+\frac{\dot{y}\dot{y}o}{\dot{x}}$, that is, CG$=\frac{\dot{x}\dot{x}+\dot{y}\dot{y}}{\dot{x}\dot{z}}$. And whereas we are at liberty to ascribe whatever velocity we please to the Fluxion of the Abscifs, to which as to an equable Fluxion the rest may be referred, make $\dot{x}=1$, and then $\dot{y}=z$, and CG$=\frac{1+zz}{\dot{z}}$; whence GD$=\frac{z+z^3}{\dot{z}}$; and DC $=\frac{\overline{1+zz}\sqrt{1+zz}}{\dot{z}}$. Therefore

Therefore any equation being propofed in which the relation of BD to AB is expreffed for defining the Curve, find the relation between $\dot{x}$ and $\dot{y}$ by PROB. I. and at the fame time fubftitute 1 for $\dot{x}$, and z for $\dot{y}$. Then from the equation that arifes by the fame PROB. I. find the relation between $\dot{x}$, $\dot{y}$, and $\dot{z}$, and at the fame time fubftitute 1 for $\dot{x}$ and z for $\dot{y}$, as before. By the former operation you will obtain the value of z, and by the latter that of $\dot{z}$, which being obtain'd produce DB to H, towards the concave part of the Curve, that it may be $DH=\frac{1+zz}{\dot{z}}$, and draw HC parallel to AB meeting the perpendicular DC in C; then will C be the Center of Curvature at the point of the Curve D. Or fince it is $1+zz=\frac{PT}{BT}$, make $DH=\frac{PT}{\dot{z}\times BT}$, or $DC=\frac{\overline{DP}|^3}{\dot{z}\times\overline{DB}|^3}$.

EXAMPLE 1. Thus the equation $ax+bx^2-y^2=0$ being propofed, (which is an equation to the Parabola whofe *Latus Rectum* is a, and *Tranfverfum* $\frac{a}{b}$,) there will arife by PROB. I. $a+2bx-2zy=0$, writing 1 for $\dot{x}$, and z for $\dot{y}$, in the refulting equation: (which otherwife fhould have been $a\dot{x}+2b\dot{x}x-2\dot{y}y=0$). Hence there arifes $2b-2zz-2\dot{z}y=0$, 1 and z being again written for $\dot{x}$ and $\dot{y}$. By the firft we have $z=\frac{a+2bx}{2y}$, and by the latter $\dot{z}=\frac{b-zz}{y}$. Therefore any point D of the Curve being given, and confequently x and y, from thence z and $\dot{z}$ will be given, which being known, make $\frac{1+zz}{\dot{z}}=GC$ or DH, and draw HC.

Or

Or if definitely you make $a=3$ and $b=1$, so that $3xx+xx=y^2$, may be the condition of the Hyperbola: if you assume $x=1$; then $y=2$, $z=\frac{5}{4}$, $\dot{z}=-\frac{9}{32}$, and $DH=-9\frac{1}{9}$. The point H being found, raise the perpendicular HC meeting the perpendicular DC before drawn, or, which is the same thing, make HD : HC : : (1 : z : :) 1 : $\frac{5}{4}$; then draw DC the Radius of Curvature.

When you think the computation will not be too prolix, you may substitute the indefinite values of z and $\dot{z}$ into $\frac{1+zz}{\dot{z}}$ the value of CG. Thus in the present example, by a due reduction you will have $DH=y+\frac{4y^3+4by^3}{aa}$. Yet the value of DH by calculation comes out negative as may be seen in the numeral example: but this only shews, that DH must be taken towards B; for if it had come out affirmative, it ought to have been drawn the contrary way.

Cor. Hence let the sign prefixed to the symbol $+b$ be changed, that it may be $ax-bxx-yy=0$. (an equation to the Ellypsis.) Then $DH=y+\frac{4y^3-4by^3}{aa}$. ——But supposing $b=0$, that the equation may become $ax-yy=0$ (an equation to the Parabola.) Then $DH=y+\frac{4y^3}{aa}$; and thence $DG=\frac{1}{2}a+2x$.

From these several expressions it may easily be concluded, that the Radius of Curvature of any Conick Section is always $\frac{\overline{4DP}|^3}{aa}$.

Ex. 2. If $x^3=ay^2-xy^2$, be proposed; (which is the equation to the Cissoid of *Diocles*) by PROB. I. it will be, first $3x^2=2az\dot{y}-2xzy-y^2$, and then

$6x=$

$6x = 2a\dot{z}y + 2azz - 2zy - 2x\dot{z}y - 2xzz - 2zy$; that is, $z = \frac{3xx + yy}{2ay - 2xy}$, and $\dot{z} = \frac{3x - azz + 2zy + xzz}{ay - xy}$. Therefore any point of the Ciſſoid being given, and thence x and y, there will be given alſo z and $\dot{z}$, which being known, make $\frac{1 + zz}{\dot{z}} = CG$.

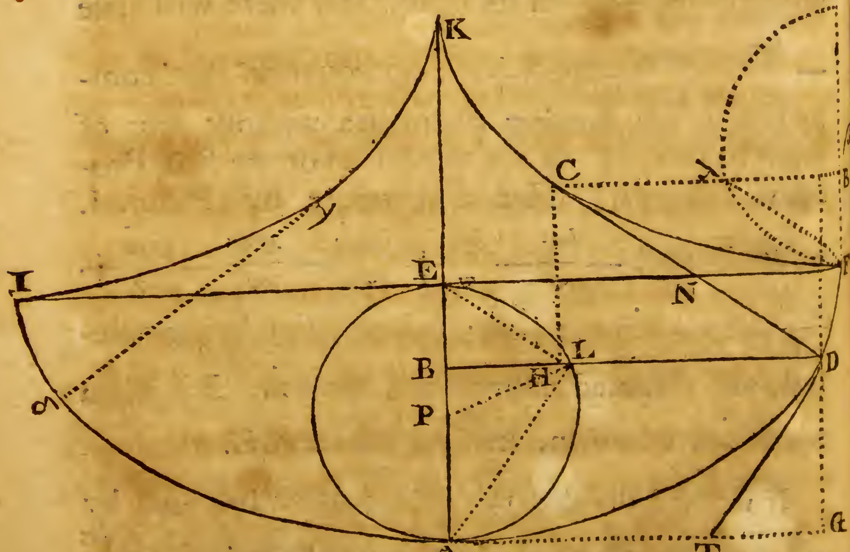
Ex. 3. If $\overline{b + y}\sqrt{cc - yy} = xy$ were given, (which is the equation to the Conchoid as before *pag.* 64.) make $\sqrt{cc - yy} = v$, and there will be $bv + yv = xy$. Now the firſt of theſe $cc - yy = vv$, will give (by PROB. I.) $-2yz = 2v\dot{v}$ (writing z for $\dot{y}$,) and the latter will give $b\dot{v} + y\dot{v} + zv = y + xz$. From theſe equations rightly diſpoſed $\dot{v}$ and z will be determined. But that $\dot{z}$ may alſo be found; out of the laſt equation exterminate the Fluxion $\dot{v}$ by ſubſtituting $-\frac{yz}{v}$ in its ſtead, and there will ariſe $-\frac{byz}{v} - \frac{yyz}{v} + zv = y + xz$, an equation that comprehends the flowing quantities without any of their Fluxions; as the reſolution of the firſt Problem requires. Hence therefore by PROB. I. we ſhall have $-\frac{bz^2}{v} - \frac{by\dot{z}}{v} + \frac{byz\dot{v}}{vv} - \frac{2yzz}{v} - \frac{yy\dot{z}}{v} + \frac{yyz\dot{v}}{vv} + \dot{z}v + z\dot{v} = 2z + x\dot{z}$; this equation being reduced, and diſpoſed in order will give $\dot{z}$. But when z and $\dot{z}$ are known, make $\frac{1 + zz}{\dot{z}} = CG$.

If we had divided the laſt equation but one by z, then by PROB. I. we ſhould have had $-\frac{bz}{v} + \frac{by\dot{v}}{vv} - \frac{2yz}{v} + \frac{yy\dot{v}}{vv} + \dot{v} = 2 - \frac{y\dot{z}}{zz}$; which would have

have been a more ſimple equation than the former for determining $\dot{z}$.

I have given this example, that it may appear, how the operation is to be performed in ſurd equations. But the curvature of the Conchoid may be thus found a ſhorter way. The parts of the equation $\overline{b+y}\sqrt{cc-yy}=xy$ being ſquared and divided by yy, there ariſes $\frac{b^2c^2}{y^2}+\frac{2bc^2}{y}\begin{matrix}+c^2\\-b^2\end{matrix}-2by-y^2=x^2$, and thence by PROB. I. $\frac{-2b^2c^2z}{y^3}\ \frac{-2bc^2z}{y^2}-2bz-2yz=2x$, or $\frac{-b^2c^2}{y^3}-\frac{bc^2}{y^2}-b-y=\frac{\dot{x}}{z}$; and hence again by PROB. I. $\frac{3b^2c^2z}{y^4}+\frac{2bc^2z}{y^3}-z=\frac{1}{z}-\frac{x\dot{z}}{zz}$, by the firſt reſult z is determined, and $\dot{z}$ by the latter.

Ex. 4. Let ADF be a Trochoid or Cycloid belonging to the circle ALE, whoſe diameter is



AE, and making the ordinate BD to cut the circle in L, call AE$=a$, AB$=x$, BD$=y$, BL$=v$, and the

the arch AL$=t$, and the Fluxion of the ſame arch $=\dot{t}$. Firſt (drawing the Semi-diameter PL) the Fluxion of the Baſe or Abſciſs AB, will be to the Fluxion of the arch AL as BL to PL; that is $\dot{x}$ or $1 : \dot{t} :: v : \frac{1}{2}a$, and therefore $\frac{a}{2v}=\dot{t}$. Then from the nature of the circle $ax-xx=vv$, and therefore by PROB. I. $a-2x=2v\dot{v}$ or $\frac{a-2x}{2v}=\dot{v}$.

Moreover from the nature of the Trochoid it is LD= arch AL, and therefore $v+t=y$, and thence by PROB. I. $\dot{v}+\dot{t}=z$. Laſtly, inſtead of the Fluxions $\dot{v}$ and $\dot{t}$, let their values be ſubſtituted, and there will ariſe $\frac{a-x}{v}=z$, whence by PROB. I. is derived $\frac{-a\dot{v}}{vv}+\frac{x\dot{v}}{vv}-\frac{1}{v}=\dot{z}$, theſe being found make $\frac{1+zz}{\dot{z}}=-$DH, and raiſe the perpendicular HC.

1. Now it follows from hence that DH$=$2BL, and CH$=$2BE, or that EF biſects the radius of curvature CD in N. This will appear by ſubſtituting the values of z and $\dot{z}$ now found, in the equation $\frac{1+zz}{\dot{z}}=$DH, and by a proper reduction of the reſult.

2. Hence the Curve FCK deſcribed indefinitely by the center of curvature of ADF is another Trochoid equal to this, whoſe vertices at I and F adjoin at the cuſpids of this. For let the circle Fλ equal and like poſited to ALE be deſcribed, and let CB be drawn parallel to EF meeting the circle in λ, then will the arch F$\lambda=$EL$=$NF$=$Cλ.

3. The line CD which is at right angles to the Trochoid IAF will touch the Trochoid IKF in the point C.

4. Hence in the inverted Trochoids, if at the cuſpid K of the upper Trochoid, a weight be hung by a thread at the diſtance KA or 2EA, and while the weight vibrates, the thread be ſuppoſed to apply itſelf to the parts of the Trochoid KF and KI, which reſiſts it on each ſide, that it may not be extended into a right line, but compel it (as it departs from the perpendicular) to be by degrees inflected above into the figure of the Trochoid, while the lower part CD from the loweſt point of contact ſtill remains a right line: the weight will move in the perimeter of the lower Trochoid, becauſe the thread CD will always be perpendicular to it.

5. Therefore the whole length KA is equal to the Perimeter of the Trochoid KCF, and its part CD is equal to the part of the Perimeter CF.

6. Since the thread by its oſcillating motion revolves about the moveable point C as a center, therefore the ſuperfices through which the whole line CD continually paſſes, will be to the ſuperfices through which the part CN above the right line IF paſſes at the ſame time, as CD*q* to CN*q*, that is, as 4 to 1. Therefore the area CFN is a fourth part of the area CFD, and the area KCNE is a fourth of the area AKCD.

7. Alſo ſince the ſubtenſe EL is equal and parallel to CN, and is turned about the immoveable center E, juſt as CN is moved about the moveable center C, the ſuperfices will be equal thro' which they paſs in the ſame time; that is, the area CFN, and the ſegment of the circle EL: and thence the area NFD will be the triple of that ſegment; and the whole area ADF will be the triple of the ſemicircle.

8. Laſtly, When the weight D arrives at the point F, the whole thread will be wound about the Trochoid KCF, and the Radius of Curvature will there be nothing. Wherefore the Trochoid IAF at the cuſpid F is more curved than any circle,

and with the Tangent BF produced makes an Angle of Contact infinitely greater than a circle can make with a right line.

There are alſo Angles of Contact, that are infinitely greater than thoſe of Trochoids, and others infinitely greater than them, and ſo on *in infinitum*; and yet the greateſt of them are infinitely leſs than Right Lined Angles.

Thus $xx=ay$, $x^3=by^2$, $x^4=cy^3$, $x^5=dy^4$, *&c.* denote a ſeries of curves, of which every ſucceeding one makes an angle with its abſciſs, which is infinitely greater than the preceding one can make with its abſciſs: The Angle of Contact which the firſt $xx=ay$ makes, is of the ſame kind with that of Circles; and that which the ſecond $x^3=by^2$ makes, is of the ſame kind with Trochoids. And tho' the Angles of the ſucceeding Curves do infinitely exceed the Angles of the preceding ones, yet they can never arrive at the magnitude of Right Lined Angles.

After the ſame manner $x=y$, $xx=ay$, $x^3=b^2y$, $x^4=c^3y$, *&c.* denote a ſeries of Lines, of which the Angles of the ſubſequents made with their abſciſſes at their vertices, are always infinitely leſs than the Angles of the preceding ones. Moreover between the Angles of Contact of any two of this kind may other Angles of Contact be found *ad infinitum*, that will infinitely exceed each other.

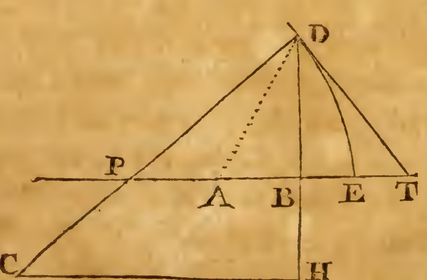
Now it appears that one kind of Angles of Contact are infinitely greater than another kind; ſince a Curve of one kind, however great it may be, cannot be interpoſed at the Point of Contact of another kind between the Curve and its Tangent, however ſmall that Curve may be; or an Angle of Contact of one kind cannot neceſſarily contain an Angle of Contact of another kind, as the whole contains a part. Thus the Angle of Contact of the curve $x^4=cy^3$, or the Angle which it makes with its

Abscis, necessarily includes the Angle of Contact of the curve $x^3=by^2$, and can never be contained by it. For Angles that mutually exceed each other are of the same kind, as it happens with the aforesaid Angles of the Trochoid, and of this Curve $x^3=by^2$.

Hence it appears that curves in some points may be infinitely more streight, or infinitely more curved, than any circle, and yet for that reason do not lose the form of curve lines. *But all this by the way only.*

Ex. 5. Let ED be the quadratrix to the circle, described from the center A, and letting fall DB perpendicular to AE, make AB$=x$, BD$=y$, and AE $=1$. Then it will be $\dot{y}x-\dot{y}y^2-\dot{y}x^2=\dot{x}y$ as before, (*pag.* 69.) Then writing 1 for $\dot{x}$, and z for $\dot{y}$, the equation becomes $zx-zy^2-zx^2=y$, thence by PROB. I. $\dot{z}x-\dot{z}y^2-\dot{z}x^2+z\dot{x}-2zx\dot{x}-2zy\dot{y}=\dot{y}$, then reducing and again writing 1 for $\dot{x}$ and z for $\dot{y}$ there arises $\dot{z}=\frac{2z^2y+2zx}{x-xx-yy}$. But z and $\dot{z}$ being found make $\frac{1+zz}{\dot{z}}=$DH, and draw HC as before.

If you desire a construction of the Problem, you will find it very short. For draw DP perpen-



dicular to DT meeting AT in P, and make 2AD : AE : : PT : CH. For $z=\frac{y}{x-xx-yy}=\frac{\text{BD}}{-\text{BT}}$, and

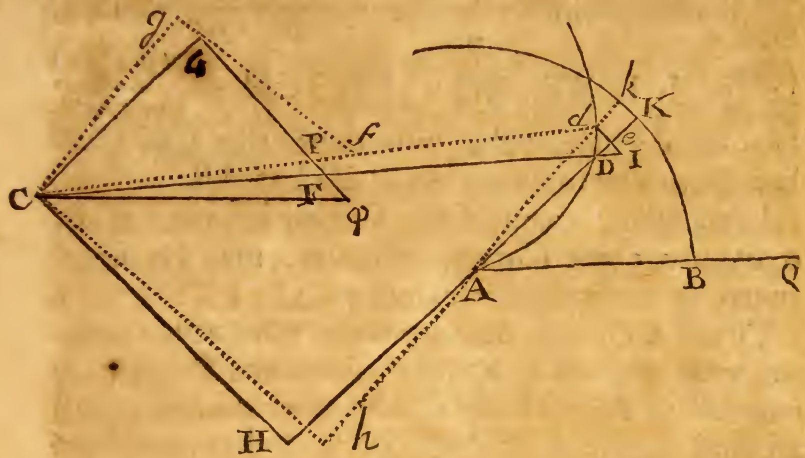
$zy=$

$zy = \frac{BDq}{-BT} = BP$. Alſo $zy + x = -AP$, and $\frac{2z}{x-xx-yy}$ into $zy + x = \frac{2BD}{AE \times BTq}$ into $-AP = \dot{z}$. Moreover it is $1 + zz = \frac{PT}{BT}$ (becauſe $= 1 + \frac{BDq}{BTq} = \frac{DTq}{BTq}$,) and therefore $\frac{1+zz}{\dot{z}} = \frac{PT \times AE \times BT}{-2BD \times AP} = DH$. Laſtly, It is $BT : BD :: DH : CH = \frac{PT \times AE}{-2AP}$. Here the negative value only ſhews that CH muſt be taken the ſame way from DH as AB.

In the ſame manner the Curvature of Spirals, or of any other Curves whatever, may be determined by a very ſhort calculation.

Furthermore, to determine the Curvature without any previous reduction, when the Curves are referred to Right Lines in any other manner, this method might have been applied, as has been done already for drawing Tangents. But as all Geometrical Curves, and alſo Mechanical ones (eſpecially when defining conditions are reduced to infinite equations as I ſhall ſhew hereafter) may be referred to rectangular Ordinates, I have done enough in this matter. He that deſires more, may eaſily ſupply it by his own induſtry; eſpecially if for a further illuſtration I ſhall add the method for Spirals. Let BK be a Circle, A its center, B a given point in its circumference. Let AD*d* be a Spiral, DC its perpendicular, and C the Center of Curvature at the point D. Then drawing the Right Line ADK, and CG parallel and equal to AK; as alſo the perpendicular GF meeting CD in F, make AB or $AK = 1 = CG$, $BK = x$, $AD = y$, and $GF = z$. Then conceive the point D to move in the Spiral for an infinitely little ſpace D*d*, and thro' *d* draw the ſemi-diameter A*k*, and C*g* parallel and equal to it, *gf* its perpendicular meeting C*d* in *f*, which

which alſo GF meets in P. Produce GF to φ, that $G\varphi = gf$, and to AK let fall the perpendicu-*de*, and produce it till it meets CD at I. Then the contemporaneous moments of BK, AD, and GF will be K*k*, D*e*, and Fφ, which therefore may be called $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$.



Now it is AK : A*e* (AD) : : *de* :*e*D$=oyz$, therefore $yz=\dot{y}$. Beſides CG : CF : : *de* : *d*D $= oy \times$ CF : : *d*D : *d*I $= oy \times \overline{CF}^2$. Moreover becauſe the Angle $PC\varphi =$ the Angle GC*g* = DA*d*, and the Angle $CP\varphi =$ the Angle C*d*I= the Angle *ed*D+ a Right Angle =AD*d*, therefore the Triangles $CP\varphi$ and AD*d* are ſimilar. And thence AD : D*d* : : CP (CF) : $P\varphi = o\overline{CF}^2$; from whence take F$\varphi$, and there will remain $PF = o \times \overline{CF}^2 - o \times \dot{z}$. Laſtly, letting fall CH perpendicular to AD, it is PF : *d*I : : CG : *e*H or $DH = \frac{y \times CFq}{CFq - \dot{z}}$. Or ſubſtituting $1+zz$ for $\overline{FC}^2$ it will be $DH = \frac{y+yzz}{1+zz-\dot{z}}$. Here it may be obſerved that in theſe kind of computations, I take thoſe quantities AD and A*e* for equal, the ratio of which differs but infinitely little from the ratio of equality.

Now

Now from hence arises the following rule. The relation of x and y being exhibited by any equation, find the relation of the Fluxions $\dot{x}$ and $\dot{y}$ by PROB. I. and substitute 1 for $\dot{x}$ and yz for $\dot{y}$. Then from the resulting equation find again, by PROB. I. the relation between $\dot{x}$, $\dot{y}$, and $\dot{z}$, and again substitute 1 for $\dot{x}$ and yz for $\dot{y}$; the first result by due reduction will give $\dot{y}$ and z, and the latter will give $\dot{z}$; which being known make $\frac{y+yzz}{1+zz-\dot{z}}=$DH, and raise the perpendicular HC, meeting the perpendicular to the Spiral DC, before drawn, in C: then C will be the center of Curvature. Or which comes to the same thing, take CH : HD : : z : 1, and draw CD.

Ex. 1. If the Equation be $ax=y$, (which will belong to the Spiral of *Archimedes*) then by PROB. I. $a\dot{x}=\dot{y}$, (or writing 1 for $\dot{x}$ and yz for $\dot{y}$) $a=yz$. Hence again by PROB. I. $0=\dot{y}z+y\dot{z}$. Wherefore any point D of the Spiral being given, and thence the length AD or y, there will be given $z=\frac{a}{y}$, and $\dot{z}=\frac{-ayz}{yy}=\frac{-az}{y}$: which being known make $1+zz-\dot{z}$: $1+zz$: : DA (y) : DH, and 1 : z : : DH : CH. Hence you will easily deduce the following construction. Produce AB to Q, so that AB : arch BK : : arch BK : BQ, and make AB+AQ : AQ : : DA : DH : a : HC.

Ex. 2. If $ax^2=y^3$ be the equation that determines the relation between BK and AD. By PROB. I. you will have $2a\dot{x}x=3\dot{y}y^2$, or $2ax=3zy^3$: Thence $2a\dot{x}=3\dot{z}y^3+9z\dot{y}y^2$. It is therefore $z=\frac{2ax}{3y^3}$ and $\dot{z}=\frac{2a-9zzy^3}{3y^3}$. This being known make

$1+zz$

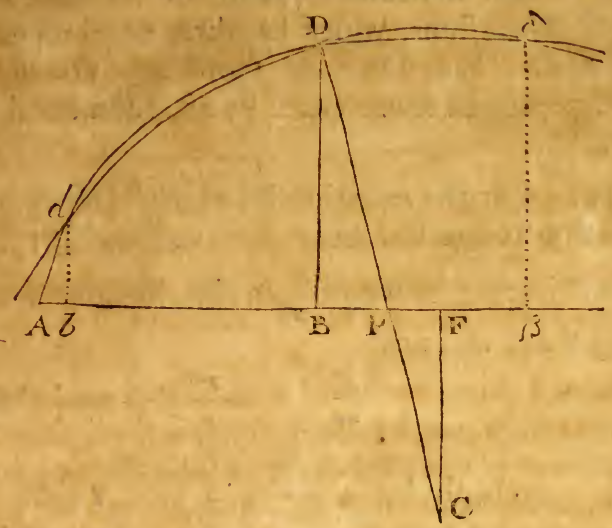
$1+zz-\dot{z} : 1+zz :: : DA : DH$, or the work being reduced to a better form make $9xx+10 : 9xx+4 :: DA : DH$.

Ex. 3. After the ſame manner, if $ax^2-bxy=y^3$, determines the relation of BK to AD, there will ariſe $\frac{2ax-by}{bxy+3y^3}=z$, and $\frac{2a-2bzy-bz^2xy-9z^2y^3}{bxy+3y^3}=\dot{z}$, from which DH, and thence the point C is determined as before.——And thus you will eaſily determine the Curvature of any other Spirals, or invent Rules for any other kinds of Curves in imitation of theſe already given.

Now I have finiſhed; but having made uſe of a method, which is pretty different from the common ways of operation; and as the Problem itſelf is of the number of thoſe which are not very frequent among Geometricians; for the illuſtration and confirmation of the Solutions here given, I ſhall not think much to give a hint of another, which is more obvious, and has a nearer relation to the uſual methods of drawing Tangents. Thus if from any center and with any radius a Circle be conceived to be deſcribed, which may cut any Curve in ſeveral points: If that Circle be ſuppoſed to be contracted or enlarged, till two of the points of interſection coincide, it will there touch the Curve: and beſides if this center be ſuppoſed to approach towards, or recede from the point of contact, till the third point of interſection ſhall meet with the former in the point of contact; then will that circle be equi-curve with the Curve in that point of contact. In like manner as inſinuated before in the laſt of the parts of the center of Curvature, by the help of which I affirmed the Problem might be reſolved in a different manner.

Therefore

Therefore with center C and radius CD let a circle be deſcribed that cuts the Curve in the points



d, D, and *δ*; and letting fall the perpendiculars DB, *db*, *δβ*, and CF to the Abſciſs AB; call AB $=x$, BD$=y$, AF$=v$, FC$=t$, and DC$=s$. Then BF$=v-x$, and DB+FC$=y+t$; the ſum of theſe Squares is equal to the ſquare of DC; that is, $v^2-2vx+x^2+y^2+2yt+t^2=s^2$. If you would abbrieviate this make $v^2+t^2-s^2=q^2$, (any ſymbol at pleaſure) and it becomes $x^2-2vx+y^2+2ty+q^2=0$. After you have found t, v, and q^2, you will have $s=\sqrt{v^2+t^2-q^2}$.

Now let any equation be propoſed for defining the Curve, the quantity of whoſe Curvature is to be found; by the help of this equation you may expunge either of the quantities *x* or *y*, and there will ariſe an equation, the roots of which, (*db*, DB, *δβ*, &c. if you exterminate *x*; or A*b*, AB, Aβ, &c. if you exterminate *y*) are at the points of interſection *d*, D, *δ*, &c. Wherefore ſince three of them become equal, the circle both touches the Curve, and will alſo be of the ſame degree of Curvature

vature as the Curve in the point of contact. But they will become equal by comparing the equation with another fictitious equation of the same number of dimensions, which has three equal roots; (as *Des Cartes* has done) or more expeditiously by multiplying its terms twice by an arithmetical progression.

Ex. Let the equation be $ax=yy$ (which is an equation to the Parabola) and expunging x (that is substituting its value $\frac{yy}{a}$ in the foregoing equation) there will arise

$$\frac{y^4}{aa} \;*\; -\frac{2v}{a}y^2 + 2ty + q^2 = 0$$

			y^2		
multiply by	4	*	2	1	0
and then by	3	*	1	0	1

and there will arise $\frac{12y^4}{aa} \;*\; -\frac{4v}{a}y^2 + 2y^2 \;*\; =0$

or $v=\frac{3y^2}{a}+\frac{1}{2}a$; whence it is easily inferred, that $BF=2x+\frac{1}{2}a$, as before.

Wherefore any point D of the Parabola being given, draw the Perpendicular DP, and in the Axis take PF=2AB, and erect the Perpendicular FC, meeting DP in C; then will C be the Center of Curvature desired. The same may be performed in the Ellypsis and Hyperbola, but the calculus is troublesome enough, and in other Curves generally very tedious.

Of Questions that have some Affinity to these.

From the resolution of this Problem some others may be performed; such are

I. *To*

I. *To find the Point where the Curve has a given Degree of Curvature.*

Thus in the Parabola $ax=yy$, if the point be required, whofe Radius of Curvature is of a given length f; from the Center of Curvature found as before, you will determine the radius to be $\frac{z+4x}{2z}\sqrt{zz+4zx}$, which muft be equal to f. Then by reduction there arifes $x=-\frac{1}{4}a+\sqrt[3]{\frac{1}{16}af^2}$.

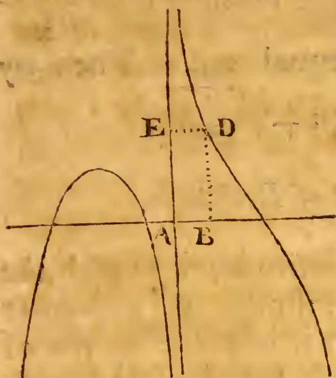
II. *To find the Point of Rectitude.*

I call that the Point of Rectitude, in which the Radius of Flexure becomes infinite, or its center at an infinite diftance. Such it is at the Vertex of the Parabola $a^3x=y^4$. And the fame Point is commonly the Limit of Contrary Flexure, whofe determination I have exhibited before. But another determination, and that not inelegant, may be derived from this Problem; which is, the longer the Radius of Flexure is, fo much the lefs the Angle DCd becomes; [*See fig. pag.* 83.] and alfo the moment δf; fo that the Fluxion of the quantity z is diminifhed along with it, and by the infinitude of that Radius altogether vanifhes. Therefore find the Fluxion $\dot{z}$, and fuppofe it to become nothing.

As if you would determine the limit of contrary Flexure in the Parabola of the fecond kind, by the help of which *Cartefius* conftructed Equations of fix Dimenfions. The Equation to that Curve is $x^3-bx^2-cdx+bcd+dxy=0$, hence by PROB. I. there arifes $3\dot{x}x^2-2b\dot{x}x-cd\dot{x}+d\dot{x}y+dx\dot{y}=0$. Now writing 1 for $\dot{x}$, and z for $\dot{y}$, and o

 for

for $\dot{z}$, it becomes $3x^2-2bx-cd+dy+dxz=0$; whence again by PROB. I. $6x\dot{x}-2b\dot{x}+d\dot{y}+d\dot{x}z+dx\dot{z}=0$: here again writing 1 for $\dot{x}$, z for $\dot{y}$, and 0 for $\dot{z}$, it becomes $6x-2b+2dz=0$, and there will arise $-cd+dy=0$, or $y=c$.



Wherefore at the Point A erect the Perpendicular $AE=c$, and through E draw ED parallel to AB, then the Point D, where it cuts the Concavo-Convex part of the Parabola, will be in the confine of Contrary Flexure.

By a like method you may determine the Points of Rectitude, which do not come between parts of Contrary Flexure. As if the Equation $x^4-4ax^3+6a^2x^2-b^3y=0$, expressed the nature of a Curve; you have first by PROB. I. $4x^3-12ax^2+12a^2x-b^3z=0$, and hence again $12x^2-24ax+12a^2-b^3\dot{z}=0$: here suppose $z=0$, and by reduction there will arise $x=a$. Wherefore take $AB=a$, [*See fig. pag.* 102.] and erect the Perpendicular BD; this will meet the Curve in the Point of Rectitude D, as was required.

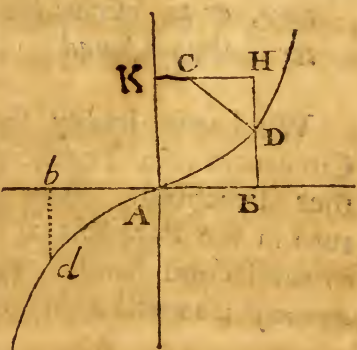
III. *To find the infinite Flexure.*

Find the Radius of Curvature, and suppose it to be equal to nothing. Thus to the Parabola of the second Kind, whose Equation is $x^3=ay^2$; that radius will be $CD=\frac{4a+9x}{6a}\sqrt{4ax+9xx}$; which becomes nothing, when $x=0$.

IV. *To determine the Point of the greatest or least Flexure.*

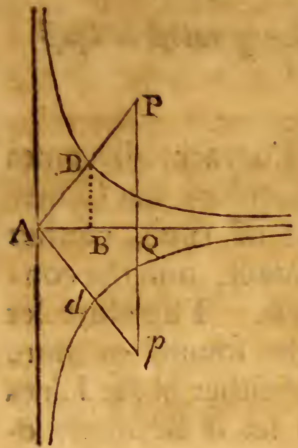
At these Points the Radius of Curvature becomes either the greatest or least; wherefore the Center of Curvature, at that moment of time, neither moves towards the Point of Contact, nor the contrary way, but is entirely at Rest. Therefore let the Fluxion of the Radius CD be found, or more expeditiously, let the Fluxion of either of the Lines AK, BH, BD, be found, and let it be made equal to nothing.

As if the Question were proposed concerning the Parabola of the second kind $x^3=a^2y$; first to determine the Center of Curvature, you will find DH $=\frac{aa+9xy}{6x}$, and therefore BH$=\frac{aa+15xy}{6x}$. Make BH$=v$, then $\frac{aa}{6x}+\frac{5}{2}y=v$; hence by PROB. I. $\frac{-a^2\dot{x}}{6xx}+\frac{5}{2}\dot{y}=\dot{v}$. Now



suppose $\dot{v}$ or the Fluxion of BH to be nothing; and besides since by Hypothesis $x^3=a^2y$, and thence (by PR. I.) $3\dot{x}x^2=a^2\dot{y}$, then putting $\dot{x}=1$ substitute $\frac{3xx}{aa}$ for $\dot{y}$, and there will arise $45x^4=a^4$. Take therefore AB$=a\sqrt[4]{\frac{1}{45}}=\frac{a}{\sqrt[4]{45}}$, and raising the Perpendicular BD, it will meet the Curve in the Point of greatest Curvature; or, which is the same thing, make AB : BD :: $3\sqrt{5}$: 1.

After



After the same manner the Hyperbola of the second kind represented by the Equation $xy^2=a^3$, will be most inflected in the Points D and d; which you may determine by taking in the Absciss $AQ=1$, and erecting the Perpendicular $QP=\sqrt{5}$, and Qp equal to it on the other side; then drawing AP and Ap, they will meet the Curve in the Points D and d required.

V. *To determine the Locus of the Center of Curvature, or to describe the Curve, in which that Center is always found.*

We have already shewn, that the Center of Curvature of a Trochoid is always found in another Trochoid. And thus the Center of Curvature in the Parabola is found in another Parabola of the second kind, represented by the Equation $axx=y^3$; as will easily appear from Calculation.

VI. *Light falling upon any Curve, to find its Focus, or the Concourse of the Rays that are refracted at any of the Points.*

Find the Curvature of that Point of the Curve, and describe a Circle, from the Center, and with the Radius, of Curvature. Then find the Concourse of the Rays, when they are refracted by a Circle about that Point; for the same is the Concourse of the refracted Rays in the proposed Curve.

To this may be added a particular Invention of the Curvature of the Vertices of Curves, where they

they cut their Bases at Right Angles. For the Point in which the Perpendicular to the Curve meeting with the Base cuts it ultimately, is the Center of its Curvature. So that having the relation between the Base or Abscifs x, and the Rectangular Ordinate y, and thence (by PROB. I.) the relation between the Fluxions $\dot{x}$ and $\dot{y}$, the value $y\dot{y}$, (if you substitute 1 for $\dot{x}$ into it, and make $y=0$) will be the Radius of Curvature.

Thus in the Ellypsis $ax-\frac{a}{b}xx=yy$, it is $\frac{a\dot{x}}{2}-\frac{a\dot{x}x}{b}=y\dot{y}$; which value of $y\dot{y}$, if we suppose $y=0$, and consequently $x=0$, writing 1 for $\dot{x}$, becomes $\frac{1}{2}a$ for the Radius of Curvature. And so at the Vertices of the Hyperbola and Parabola, the Radius of Curvature will be always half of the *Latus Rectum*.

In like manner for the Conchoid defined by the Equation $\frac{b^2c^2}{xx}+\frac{2bcc}{x}\begin{matrix}+cc\\-bb\end{matrix}-2bx-xx=yy$, the value of $y\dot{y}$ found by PROB. I. will be $\frac{-b^2c^2}{x^3}-\frac{bc^2}{x^2}-b-x$. Now suppose $y=0$, and thence $x=c$ or $-c$, we shall have $\frac{-bb}{c}-2b-c$, or $\frac{bb}{c}-2b+c$, for the Radius of Curvature. Therefore make AE : EG :: EG : EC, [*See fig. pag.* 65] and Ae : eG :: eG : ec, and you will have the Centers of Curvature C and c at the Vertices of the Conjugate Conchoids E and e.

PROBLEM

Problem VI.

To determine the Quality of the Curvature at a given Point of any Curve.

By the Quality of Curvature, I mean its Form as it is more or less inequable, or as it is more or less varied in its progress through different parts of the Curve. So if it were demanded, what is the Quality of the Curvature of the Circle? It might be answered, that it is uniform or invariable. And thus if it were demanded what is the Quality of the Curvature of the Spiral, which is described by the motion of the point D, [*See fig. pag.* 94.] proceeding from A in AD with an accelerated Velocity, while the line AK moves with an uniform Rotation about the Center A; the acceleration of which velocity is such, that the Right Line AD has the same ratio to the Arch BK, described by a given point K, as a Number has to its Logarithm. I say, if it be asked what is the Quality of the Curvature of this Spiral? It may be answered, that it is uniformly varied, or that it is equably inequable. And thus other Curves in their several points may be denominated inequably inequable, acccording to the variation of their Curvature.——Therefore the Inequability (or Variation) of Curvature is required at any point of a Curve. *Concerning which it may be observed.*

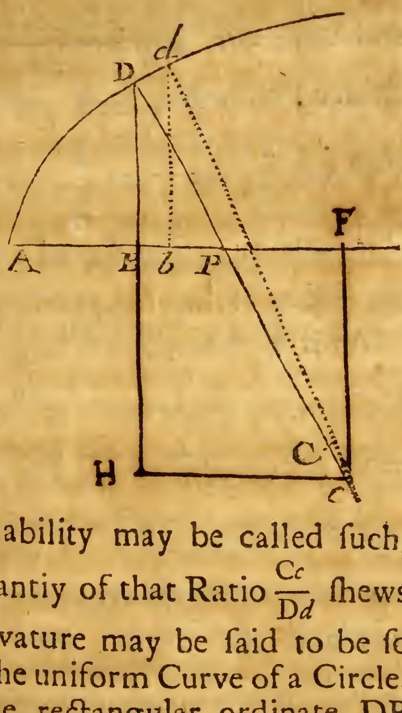
1. That at the Points which in similar Curves are alike posited, there is a like inequability or variation of Curvature.

2. That the moments of the Radii of Curvature at these Points are proportional to the contemporaneous moments of the Curves, and the Fluxions to the Fluxions.

3. And

3. And therefore that when these Fluxions are not proportional, the Inequability of the Curvature will be dissimilar: for there will be a greater inequability where there is a greater ratio of the Fluxion of the Radius of Curvature to the Fluxion of the Curve. And therefore that Ratio of the Fluxion may not improperly be called the *Index* of the Inequability, or of the Variation, of Curvature.

At the Points D, and *d*, infinitely near to each other in the Curve AD*d*, let there be drawn the Radii of Curvature DC and *dc*, then D*d* being the moment of the Curve, C*c* will be the contemporaneous moment of the Radius of Curvature, and $\frac{Cc}{Dd}$ will be the Index of the Inequability of that Curvature; for the Inequability may be called such, and so great, as the quantiy of that Ratio $\frac{Cc}{Dd}$ shews it to be. Or the Curvature may be said to be so much more unlike to the uniform Curve of a Circle.



Now letting fall the rectangular ordinate DB and *db* to any Line AB, meeting DC in P, make AB $=x$, BD $=y$, DP $=t$, DC $=v$, thence it will be B*b* $=\dot{x}o$, and C*c* $=\dot{v}o$; and BD : DP : : B*b* : D*d* $=\frac{\dot{x}t}{y}$; and $\frac{Cc}{Dd}=\frac{\dot{v}y}{\dot{x}t}=\frac{\dot{v}y}{t}$ making $\dot{x}=1$. Wherefore the relation between x and y being exhibited by any Equation, and thence according to PROB. IV. and V. the Perpendicular DP or t being found, and the Radius of Curvature v, and the Fluxion $\dot{v}$

of that Radius by PROB. I. the Index $\frac{\dot{v}y}{t}$ of the Inequability of Curvature will be given also.

EXAMPLE I. Let the Equation to the Parabola $2ax=yy$ be given. Then by PROB. IV. $BP=a$, and therefore $DP=\sqrt{aa+yy}=t$. Also by PROB. V. $BF=a+2x$, and $BP : DP :: BF : DC=\frac{at+2tx}{a}=v$. These Equations by PROB. I. give $2a\dot{x}=2\dot{y}y$, and $2\dot{y}y=2\dot{t}t$, and $\frac{a\dot{t}+2\dot{t}x+2t\dot{x}}{a}=\dot{v}$, which being reduced to order, and putting $\dot{x}=1$, there will arise $\dot{y}=\frac{a}{y}$, $\dot{t}=\frac{\dot{y}y}{t}=\frac{a}{t}$, and $\dot{v}=\frac{a\dot{t}+2\dot{t}x+2t}{a}$. Thus $\dot{y}$, $\dot{t}$, and $\dot{v}$ being found, there will be had $\frac{\dot{v}y}{t}$; the Index of the Inequability of Curvature.

As if in Numbers it were determined that $a=1$, or $2x=yy$; and $x=\frac{1}{2}$. Then $y=\sqrt{2x}=1$, $\dot{y}=\frac{a}{y}=1$, $t=\sqrt{aa+yy}=\sqrt{2}$, $\dot{t}=\frac{a}{t}=\sqrt{\frac{1}{2}}$, and $\dot{v}=(\frac{a\dot{t}+2\dot{t}x+2t}{a}=)\ 3\sqrt{2}$. So that $\frac{\dot{v}y}{t}=3$; which therefore is the Index of Inequability. But if it were determined that $x=2$, then $y=2$, $\dot{y}=\frac{1}{2}$, $t=\sqrt{5}$, $\dot{t}=\sqrt{\frac{1}{5}}$, and $\dot{v}=3\sqrt{5}$; that is, $\frac{\dot{v}y}{t}=6$ will be here the Index of Inequability.

Therefore the inequability of Curvature at that point of the Curve, from whence letting fall an Ordinate it will be equal to the *Latus Rectum* of the Parabola, will be double to the Inequability at that point from whence the Ordinate is $\frac{1}{2}$ of the *Latus Rectum*; that is, the Curvature in that Point is as unlike again to the Curvature of the Circle, as the Curvature at the second Point.

Let

Let the Equation be $2ax-bxx=yy$. By PROB. IV. it will be $a-bx=$BP, and thence $aa-2abx+b^2x^2+y^2=t^2$, or $aa-byy+yy=tt$. Alſo by PROB. V. it is DH$=y+\frac{y^3-by^3}{aa}$; where if for $yy-byy$ you ſubſtitute $tt-aa$, there ariſes DH$=\frac{tty}{aa}$. It is alſo BD : DP : : DH : DC$=\frac{t^3}{a^2}=v$. Now by PROB. I. the Equations $2ax-bxx=yy$, $aa-byy+yy=tt$, and $\frac{t^3}{aa}=v$, give $a-bx=\dot{y}y$, and $\dot{y}y-b\dot{y}y=\dot{t}t$, and $\frac{3t^2\dot{t}}{aa}=\dot{v}$; thus $\dot{v}$ being found, $\frac{\dot{v}y}{t}$ the Index of the Inequability of Curvature will alſo be known.

Thus in the Ellypſis $2x-3xx=yy$; where it is $a=1$, $b=3$; if we make $x=\frac{1}{2}$, then $y=\frac{1}{2}$, $\dot{y}=-1$, $t=\sqrt{\frac{1}{2}}$, $\dot{t}=\sqrt{2}$, $\dot{v}=3\sqrt{\frac{1}{2}}$; therefore $\frac{\dot{v}y}{t}=\frac{3}{2}$, which is the Index of the Inequability of Curvature. Hence it appears that the Curvature of this Ellypſis, at the point D here aſſumed, is Two times leſs inequable, (or Two times more like to the Curvature of the Circle) than the Curvature of the Parabola at that Point of its Curve, from whence an Ordinate let fall upon the Axis is equal to half the *Latus Rectum*.

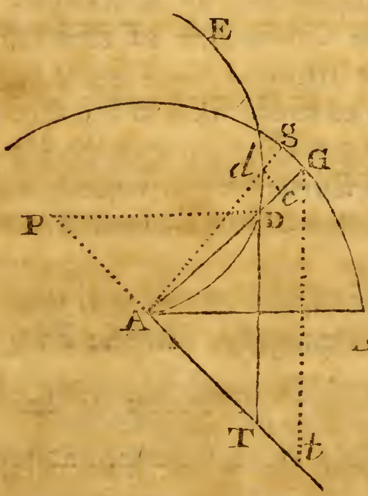
If we have a mind to compare the ſeveral concluſions obtained in theſe examples. In the Parabola $2ax=yy$ ariſes $\frac{\dot{v}y}{t}=\frac{3y}{a}$ for the Index of Inequability; in the Ellypſis, $2ax-bx=yy$, ariſes $\frac{\dot{v}y}{t}=\frac{3y-3by}{aa}\times$BP, and ſo in the Hyperbola $2ax+bx=yy$, (the analogy being obſerved) there ariſes the Index $\frac{\dot{v}y}{t}=\frac{3y+3by}{aa}\times$BP. Hence it is found, that at the different points of any Conick Section con-

considered apart, the Inequability of Curvature is as the rectangle BD × BP: And that at the several Points of the Parabola it is as the Ordinate BD.

Now as the Parabola is the most simple Figure of those, that are curved with an Inequable Curvature, and as the Inequability of its Curvature is so easily determined; (for its Index is $\frac{6 \times Ordinate}{Latus\ Rectum}$) therefore the Curvature of other Curves may not improperly be compared to the Curvature of this.

As if it were inquired what may be the Curvature of the Ellypsis $2x - 3xx = yy$, at that Point of the perimeter, which is determined by assuming $x = \frac{1}{2}$; because its Index is $\frac{3}{2}$ as before, it might be answered, that it is like the Curvature of the Parabola $6x = yy$, at that Point of the Curve, between which and the Axis the Ordinate is equal to $\frac{3}{2}$.

Thus as the Fluxion of the Spiral ADE before described is to the Fluxion of the subtense AD in a certain given ratio, suppose as d to e. On its concave side erect $AP = \frac{e}{\sqrt{dd - ee}} \times AD$ perpendicular to AD, then P will be the Center of Curvature; and $\frac{AP}{AD}$ or $\frac{e}{\sqrt{dd - ee}}$ will be the Index of Inequability. So that this Spiral has every where its Curvature alike Inequable, in the same form as the Parabola $6x = yy$ in that point of its Curve, from whence to its Abscis or Base a perpendicular Ordinate is let fall, which is equal to the quantity $\frac{e}{\sqrt{dd - ee}}$.



And

And thus the Index of Inequability at any Point D of the Trochoid [*See fig. pag.* 88.] is found to be $\frac{AB}{BL}$. Wherefore its Curvature at the same Point D is as inequable, or as unlike to that of the Circle, as the Curvature of any Parabola $ax=yy$ is, at the Point where the Ordinate is $\frac{1}{6}a=\frac{AB}{BL}$.

From these considerations the sense of the Problem (I conceive) must be plain enough; which being well understood it will not be difficult for any one, who observes the Series of the things above delivered, to furnish himself with more examples; and to contrive many other methods of operation, as occasion may require. So that he will be able to manage Problems of a like nature (where he is not discouraged by a tedious and perplexed calculation) with little or no difficulty. Such are these following.

I. *To find the Point where there is either no Inequality of Curvature; or infinite; or the greatest; or the least. Thus at the Vertices of the Conick Sections there is no Inequability of Curvature. At the Cuspid of the Trochoid it is infinite. And it is greatest at that Point of the Ellypsis, where the Rectangle* BD × BP [*See fig pag.* 105.] *is greatest; that is, where the Diagonal Lines of the circumscribed Parallelogram cut the Ellypsis, whose sides touch it in the Principal Vertices*

II. *To determine a Curve of some Definite Species, (suppose a Conick Section,) whose Curvature at any Point may be Equal and Similar to the Curvature of any other Curve at a given Point of it.*

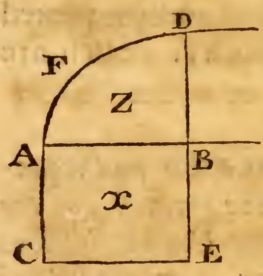
III. *To determine a Conick Section, at any point of which the Curvature and Position of the Tangent in respect of the Axis, may be like to the Curvature and Position of the Tangent at a Point found of any other*

other Curve. The Use of this Problem is this, that instead of Ellypses of the second Kind, whose properties of refracting light are explained by Des Cartes *in his Geometry, Conick Sections may be substituted, which will perform the same thing very near as to their refraction. And the same may be understood of other Curves.*

PROBLEM VII.

To find as many Curves as you please, whose Areas may be exhibited by finite Equations.

Let AB be the Abscis of a Curve, at whose Vertex A, let the Perpendicular AC $=1$ be raised, and let CE be drawn parallel to AB. Let also DB be a Rectangular Ordinate, meeting the Right Line CE in E, and the Curve AD in D. And conceive these Areas ACEB and ADB to be generated by the Right Lines BE and BD, as they move along the line AB.



Then their Increments or Fluxions will be also as the described lines BE and BD. Wherefore make the Parallelogram ACEB or AB $\times 1 = x$, and the Area of the Curve ADB call z; then the Fluxions $\dot{x}$ and $\dot{z}$ will be as BE and BD, so that make $\dot{x} = 1 =$ BE, then $\dot{z} =$ BD.

Now if any equation be assumed at pleasure for determining the relation between z and x, from thence by PROB. I. may $\dot{z}$ be derived. Thus there will be two Equations; the latter of which will determine the Curve; and the former its Area.

EXAMPLES,

EXAMPLES.

Assume $xx=z$, thence by PROB. I. $2x\dot{x}=\dot{z}$, or $2x=\dot{z}$, because $\dot{x}=1$.

Assume $\frac{x^3}{a}=z$, thence will arise $\frac{3x^2}{a}=\dot{z}$, an Equation to the Parabola.

Assume $ax^3=zz$ or $a^{\frac{1}{2}}x^{\frac{3}{2}}=z$, and there arises $\frac{3}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}=\dot{z}$, or $\frac{9}{4}ax=\dot{z}\dot{z}$, an Equation again to the Parabola.

Assume $a^6x^{-2}=zz$, or $a^3x^{-1}=z$, and there arises $-a^3x^{-2}=\dot{z}$, or $a^3+\dot{z}xx=0$. Here the negative value of $\dot{z}$ only insinuates that BD is to be taken the contrary way from BE.

Again if you assume $c^2a^2+c^2x^2=z^2$, you will have $2c^2x=2z\dot{z}$, and z being exterminated there will arise $\frac{cx}{\sqrt{aa+xx}}=\dot{z}$.

Or if you assume $\frac{aa+xx}{b}\sqrt{aa+xx}=z$, make $\sqrt{aa+xx}=v$, and it will be $\frac{v^3}{b}=z$, then by PROB. I. $\frac{3vv\dot{v}}{b}=\dot{z}$. Also the equation $aa+xx=vv$, gives $2x=2v\dot{v}$, by the help of which, if you expunge $\dot{v}$, it will become $\frac{3vx}{b}=\dot{z}=\frac{3x}{b}\sqrt{aa+xx}$.

Or if you assume $8-3xz+\frac{2}{5}z=zz$; you will obtain $-3z-3x\dot{z}+\frac{2}{5}\dot{z}=2z\dot{z}$. Wherefore by the assumed equation first seek the Area z; and the Ordinate $\dot{z}$ by the resulting equation.

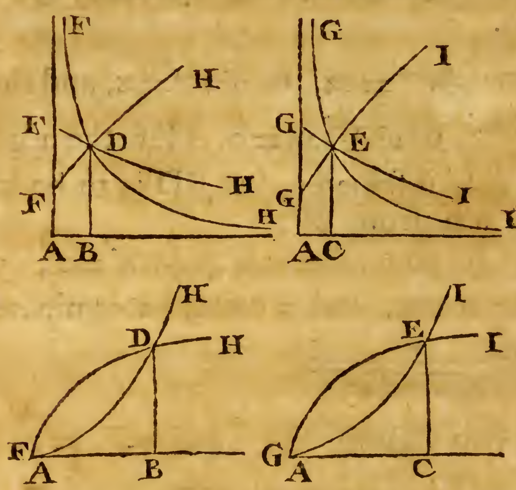
And thus from the Areas which way soever found, you may always determine the Ordinate to which they belong.

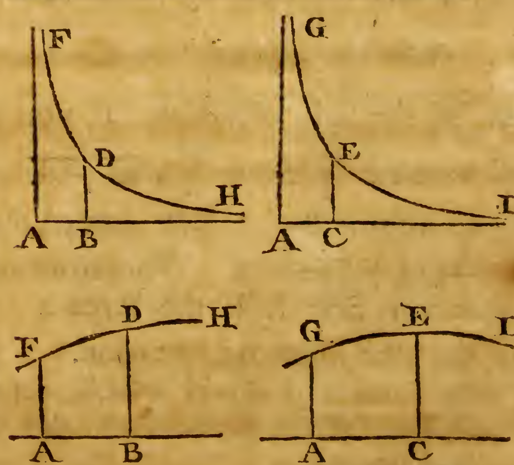
PROBLEM

PROBLEM VIII.

To find as many Curves as you please, whose Areas will have a relation to the Area of any given Circle, assignable by finite equations.

Let FDH be a given Curve, and GEI a Curve





required, and conceive their Ordinates to move at right angles upon their Abſciſſes or Baſes AB and AC; then the Increment or Fluxions of the Areas which they deſcribe, will be as thoſe Ordinates drawn into their velocities of moving, that is, into the Fluxions of their Abſciſſes. Therefore make $AB=x$, $BD=v$, $AC=z$, and $CE=y$. The Area $AFDB=s$, and the Area $AGEC=t$, and let the Fluxions of the Areas be $\dot{s}$ and $\dot{t}$; then it will be $\dot{x}v : \dot{z}y :: \dot{s} : \dot{t}$. Therefore if we ſuppoſe $\dot{x}=1$, and $v=\dot{s}$ as before, it will be $\dot{z}y=\dot{t}$, and thence $\frac{\dot{t}}{\dot{z}}=y$.

Therefore let any two Equations be aſſumed, one of which may expreſs the relation of the Areas s and t, and the other the relation of their Abſciſſes x and z, and thence by PROB. I. let the Fluxions $\dot{x}$ and $\dot{z}$ be found, and then make $\frac{\dot{t}}{\dot{z}}=y$.

EXAMPLE I. Let the given Curve AFD be a Circle expreſſed by the equation $ax-xx=vv$, and let other Curves be ſought, whoſe Areas may be equal to that of the Circle: therefore by Hypotheſis $s=t$, and thence $\dot{s}=\dot{t}$, and $y=\frac{\dot{t}}{\dot{z}}=\frac{v}{\dot{z}}$. It remains to determine $\dot{z}$ by aſſuming ſome relation between the Abſciſſes x and z.

As if you ſuppoſe $ax=zz$; then by PROB. I. $a=2\dot{z}z$; ſo that ſubſtituting $\frac{a}{2z}$ for $\dot{z}$, then $y=\frac{v}{\dot{z}}=\frac{2vz}{a}$. But it is $v=(\sqrt{ax-xx}=)\ \frac{z}{a}\sqrt{aa-zz}$; therefore $\frac{2zz}{aa}\sqrt{aa-zz}=y$ is the Equation to the Curve whoſe Area is equal to that of the Circle.

After the same manner, if you suppose $xx = z$, there will arise $2x = \dot{z}$, and thence $y = (\frac{v}{\dot{z}} =) \frac{v}{2x}$; whence v and x being exterminated, it will be

$$= \frac{\sqrt{az^{\frac{1}{2}} - z}}{2z^{\frac{1}{2}}}.$$

If you suppose $cc = xz$, there arises $0 = z + x\dot{z}$, and thence $\frac{-vx}{z} = y = \frac{c^3}{z^3}\sqrt{az - cc}$.

Again suppose $ax + \frac{s}{1} = z$, by PROB. I. it is $a + \dot{s} = \dot{z}$, and thence $\frac{v}{a + \dot{s}} = y = \frac{v}{a + v}$; which denotes a Mechanical Curve.

EX. 2. Let the Circle $ax - xx = vv$, be given again, and let Curves be sought, whose Areas may have any other assumed relation to the Area of the Circle. As if you assume $cx + s = t$, and suppose also $ax = zz$; by PROB I. it is $c + \dot{s} = \dot{t}$, and $a = 2\dot{z}z$; therefore $y = \frac{\dot{t}}{\dot{z}} = \frac{2cz + 2\dot{s}z}{a}$; and substituting $\sqrt{ax - xx}$ for $\dot{s}$, and $\frac{zz}{a}$ for x, it is $y = \frac{2cz}{a} + \frac{2zz}{aa}\sqrt{aa - zz}$.

But if you assume $s - \frac{2v^3}{3a} = t$, and $x = z$, you will have $\dot{s} - \frac{2\dot{v}v^2}{a} = \dot{t}$, and $1 = \dot{z}$; therefore $y = \frac{\dot{t}}{\dot{z}} = \dot{s} - \frac{2\dot{v}v^2}{a}$, or $= v - \frac{2\dot{v}v^2}{a}$. Now for expunging $\dot{v}$, the Equation $ax - xx = vv$ gives by PROB. I. $a - 2x = 2\dot{v}v$, and therefore $y = \frac{2vx}{a}$; where if you ex-

punge

punge v and x by substituting their values $\sqrt{ax-xx}$, and z, there will arise $y=\frac{2z}{a}\sqrt{az-zz}$.

But if you assume $ss=t$, and $x=zz$, there will arise $2\dot{s}s=\dot{t}$, and $1=2\dot{z}z$, and therefore $y=\frac{\dot{t}}{\dot{z}}=4\dot{s}sz$; then for $\dot{s}$ and x substituting $\sqrt{ax-xx}$ and zz, it will become $y=4szz\sqrt{a-zz}$, which is an equation to a Mechanical Curve.

Ex. 3. After the same manner Figures may be found, which have any assumed relation to any other given Figures. Let the Hyperbola $cc+xx=vv$ be given; then if you assume $s=t$, and $xx=cz$, you will have $\dot{s}=\dot{t}$, and $2x=c\dot{z}$; and thence $y=\frac{\dot{t}}{\dot{z}}=\frac{c\dot{s}}{2x}$. Then substituting $\sqrt{cc+xx}$ for $\dot{s}$, and $c^{\frac{1}{2}}z^{\frac{1}{2}}$ for x, it will be $y=\frac{c}{2z}\sqrt{cz+zz}$.

And thus if you assume $xv-s=t$, and $xx=cz$; you will have $v+\dot{v}x-\dot{s}=\dot{t}$ and $2x=c\dot{z}$; but $v=\dot{s}$, and thence $\dot{v}x=\dot{t}$; therefore $y=\frac{\dot{t}}{\dot{z}}=\frac{c\dot{v}}{2}$. But now by PROB. I. $cc+xx=vv$ gives $x=\dot{v}v$, and it is $y=\frac{cx}{2v}$; then substituting $\sqrt{cc+xx}$ for v, and $c^{\frac{1}{2}}z^{\frac{1}{2}}$ for x it becomes $y=\frac{cz}{2\sqrt{cz+zz}}$.

Ex. 4. Moreover if the Cissoid $\frac{xx}{\sqrt{ax-xx}}=v$ were given, to which other related Figures are to be found; and for that purpose you assume $\frac{x}{3}\sqrt{ax-xx}+\frac{2}{3}s=t$; suppose $\frac{x}{3}\sqrt{ax-xx}=b$, and its

 Fluxion

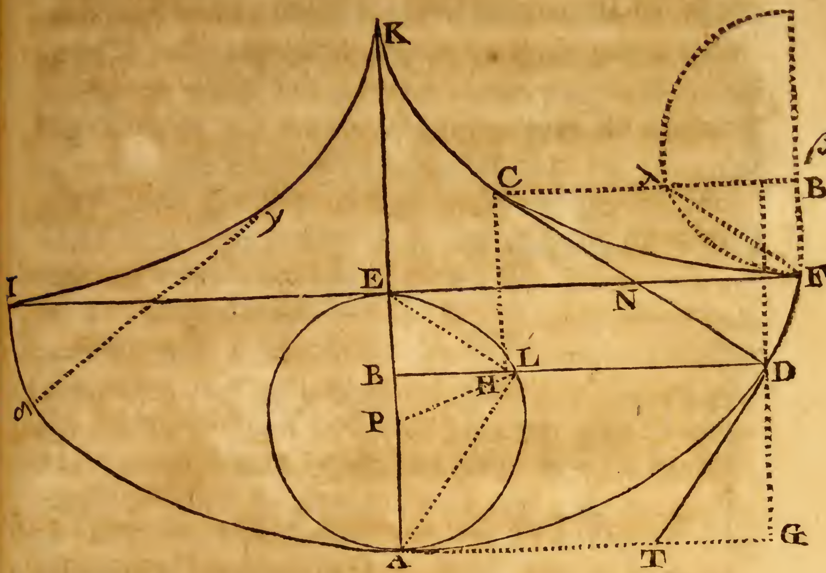
Fluxion to be $\dot{h}$; therefore $\dot{h}+\frac{2}{3}\dot{s}=\dot{t}$, but the equation $\frac{ax^3-x^4}{9}=hh$ gives $\frac{3ax^2-4x^3}{9}=2\dot{h}h$; where if you exterminate h, it will be $\dot{h}=\frac{3ax-4xx}{6\sqrt{ax-xx}}$; and besides since it is $\frac{2}{3}\dot{s}=\frac{2}{3}v=\frac{4xx}{6\sqrt{ax-xx}}$, it will be $\frac{ax}{2\sqrt{ax-xx}}=\dot{t}$. Now to determine z and $\dot{z}$, assume $\sqrt{aa-xx}=z$, and then by PROB. I. $-a=2\dot{z}z$, or $\dot{z}=\frac{-a}{2z}$. Wherefore it is $y=\frac{\dot{t}}{\dot{z}}=\frac{-zx}{\sqrt{ax-xx}}=\sqrt{\frac{2zx}{a-x}}$ $=\sqrt{ax}=\sqrt{aa-zz}$; and as this equation belongs to the Circle, we shall have the relation of the Areas of the Circle and the Cissoid.

Thus if you had assumed $\frac{2x}{3}\sqrt{ax-xx}+\frac{1}{3}s=t$, and $x=z$, there would have been derived $y=\sqrt{az-zz}$, an Equation again to the Circle.

In like manner, if any Mechanical Curve were given, other Mechanical Curves related to it might have been found. But to derive Geometrical Curves, it will be convenient, that of Right Lines depending geometrically on each other, some one may be taken for the Base or Abscis; and that the Area, which compleats the parallellogram, be sought, by supposing its Fluxion to be equivalent to the Abscis drawn into the Fluxion of the Ordinate.

Ex. 5. Thus the Trochoid ADF being proposed, I refer it to the Abscis AB, and the parallellogram ABDG being compleated, I seek for the complemental Superficies ADG, by supposing it to be described by the motion of the right line GD drawn into the velocity of the motion, that is $x\times\dot{v}$. Now whereas AL is parallel to the Tangent

gent DT: Therefore AB will be to BL, as the Fluxion of the ſame AB to the Fluxion of the



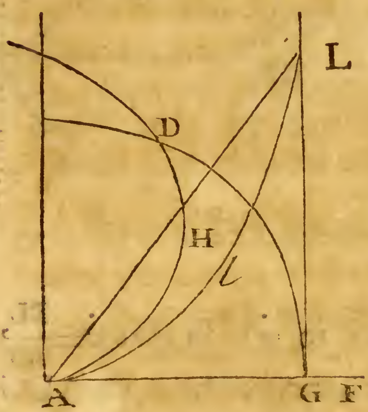
Ordinate BD, that is, as 1 to $\dot{v}$; ſo that $\dot{v}=\frac{BL}{AB}$, and therefore $x\dot{v}=BL$. Therefore the Area ADG is deſcribed by the Fluxion BL; and ſince the Circular Area ALB is deſcribed by the ſame Fluxion they will be equal.

In like manner if you conceive ADF to be a Figure of Arches, or of verſed ſines, that is, whoſe Ordinate BD is equal to the Arch AL. Since the Fluxion of the Arch AL is to the Fluxion of the Abſciſs AB, as PL to BL, that is $\dot{v} : 1 :: \frac{1}{2}a : \sqrt{ax-xx}$, then $\dot{v}=\frac{a}{2\sqrt{ax-xx}}$: and $\dot{v}x$ the Fluxion of the Area ADG will be $\frac{ax}{2\sqrt{ax-xx}}$. Wherefore if a Right Line $=\frac{ax}{2\sqrt{ax-xx}}$ be conceived to be applied as a rectangular Ordinate at B, a point of the line AB, it will be terminated at a certain Geo-

Geometrical Curve, whose Area adjoining to the Abscifs AB is equal to the Area ADG.

And thus Geometrical Figures may be found, equal to other Figures made by the application (in any angle) of arches of a Circle, of an Hyperbola, or of any other Curve, to the Sines, right or versed, of those arches; or to any other Right Lines, that may be geometrically determined.

As to Spirals the matter will be very short. For from the Center of Rotation A the arch DG that meets AF in G, and the Spiral in D; since that arch like a line moving upon the Abscifs AG describes the Area of the Spiral AHDG, so that the Fluxion of that Area is to the Fluxion of the Rectangle 1 × AG, as the Arch GD to 1; if you raise the perpendicular right line GL equal to that arch, this by moving in like manner upon the same AG, will describe the Area ALG equal to the Area of the Spiral AHDG, the Curve A*l*L being a Geometrical Curve.



And further, if the Subtense AL be drawn, then the Triangle ALG $=\frac{1}{2}$AG × GL $=\frac{1}{2}$AG × GD = Sector AGD. Therefore the complemental segments AL*l*, ADH, will also be equal. And this agrees not only to the Spiral of *Archimedes*, (in which case A*l*L becomes the Parabola of *Apollonius*;) but to any other whatsoever; that is, all of them may be converted into equal Geometrical Curves with the same ease.

I might have produced more Specimens of the Construction of this Problem, but these may suffice,

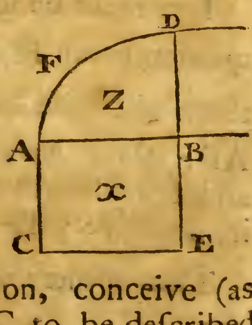
fice, as being ſo general, that whatever has yet been found out concerning the Areas of Curves, or (I believe) can be found out, is in ſome manner contained herein, and is here determined with leſs trouble, and without the uſual perplexities.

But the chief uſe of this, and the foregoing Problems is, that aſſuming the Conick Sections, or any other Curves of any known magnitude, other Curves may be found out that may be compared with theſe; and that their defining equations may be diſpoſed orderly in a Catalogue or Table. That after ſuch a Table is conſtructed, when the Area of any Curve is to be found; if its defining equation may either be found in the Table, or may be transformed into another, that is contained in the Table; then the Area may be known. Moreover ſuch a Catalogue or Table may be applied to the determining of the Lengths of Curves; to the finding of their Centers of Gravity; their Solids by their Rotation; the Superficies of thoſe Solids; or to the finding of any other Flowing Quantities produced by a Fluxion analogous to it.

PROBLEM IX.

To determine the Area of any Curve propoſed.

The Reſolution of the Problem depends upon this; that from the relation of the Fluxions being given, the relation of the Fluents may be found, as in PROB. II. Firſt if the Right Line BD, by the motion of which the Area required AFDB is diſcribed, move upright upon an Abſciſs or Baſe AB given in poſition, conceive (as before) the parallelogram ABEC to be deſcribed in the mean time on the other ſide BE by a Line



equal

equal to 1, and BE being ſuppoſed equal to the Fluxion of the Parallelogram, BD will be the Fluxion of the Area required.

Therefore make AB$=x$, then ABEC$\times 1 = 1 \times x = x$, and BE$=\dot{x}$, call AFDB$=z$, and it will be BD$=\dot{z}$, as alſo $\frac{\dot{z}}{\dot{x}}$, becauſe $\dot{x}=1$; therefore by the equation expreſſing BD, at the ſame time the ratio of the Fluxion $\frac{\dot{z}}{\dot{x}}$ is expreſſed, and thence (by PROB. II. *Caſe* I.) may be found the relation of the Flowing Quantities x and z.

EXAMPLE I. *When* BD *or* $\dot{z}$ *is equivalent to ſome ſimple Quantity.*

Let there be given $\frac{xx}{a}=\dot{z}$, or $\frac{\dot{z}}{\dot{x}}$, the equation to the Parabola; and (by PROB. II.) there will ariſe $\frac{x^3}{3a}=z$; therefore $\frac{x^3}{3a}$, or $\frac{1}{3}$ AB $\times$ BD is equal to the Area of the Parabola AFDB.

Let there be given $\frac{x^3}{aa}=\dot{z}$, an equation to the Parabola of the ſecond kind, and there will ariſe $\frac{x^4}{4a^2}=z$; that is $\frac{1}{4}$ AB $\times$ BD$=$ area AFDB.

Let there be given $\frac{a^3}{xx}=\dot{z}$, or $a^3x^{-2}=\dot{z}$, an equation to an Hyperbola of the ſecond kind, and there will ariſe $-a^3x^{-1}=z$, or $\frac{-a^3}{x}=z$; that is AB $\times$ BD$=$ Area HDBH [*See Fig. pag.* 124.] of an infinite length on the other ſide the Ordinate, as its negative value intimates.

Thus if there were given $\frac{a^4}{x^3}=\dot{z}$, there will ariſe $\frac{-a^4}{2xx}=z$.

Moreover

Moreover, let $a\dot{x}=z\dot{z}$, or $a^{\frac{1}{2}}x^{\frac{1}{2}}=\dot{z}$; (an equation again to the Parabola,) and there will arise $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}=z$; that is $\frac{2}{3}$AB×BD= Area AFDB.

Let $\frac{a^3}{x}=z\dot{z}$; then is $2a^{\frac{3}{2}}x^{\frac{1}{2}}=z$, or 2AB×BD =AFDH.

Let $\frac{a^5}{x^3}=z\dot{z}$; then $\frac{-2a^{\frac{5}{2}}}{x^{\frac{1}{2}}}=z$, or 2AB×BD= HDBH.

Let $ax^2=\dot{z}^3$; then $\frac{3}{5}a^{\frac{1}{3}}x^{\frac{5}{3}}=z$; or $\frac{3}{5}$AB×BD= AFDH. And so in others.

Ex. 2. *Where $\dot{z}$ is equal to an aggregate of such quantities.*

Let $x+\frac{xx}{a}=\dot{z}$; then $\frac{xx}{2}+\frac{xxx}{3a}=z$.

Let $a+\frac{a^3}{xx}=\dot{z}$; then $ax-\frac{a^3}{x}=z$.

Let $3x^{\frac{1}{2}}-\frac{5}{xx}-\frac{2}{x^{\frac{1}{2}}}=\dot{z}$; then $2x^{\frac{3}{2}}+\frac{5}{x}-4x^{\frac{1}{2}}$ $=z$.

Ex. 3. *Where a previous Reduction by Division is required.*

Let there be given $\frac{aa}{b+x}=\dot{z}$, (an equation to the *Apollonian* Hyperbola) and the division being performed *in infinitum*, it will be $\dot{z}=\frac{aa}{b}-\frac{aax}{b^2}+\frac{aax^2}{b^3}-\frac{aax^3}{b^4}$, &c. And thence (by PROB. II. as in Ex. 2.) you will obtain $z=\frac{a^2x}{b}-\frac{a^2x^2}{2b^2}+\frac{a^2x^3}{3b^3}-\frac{a^2x^4}{4b^4}$, &c.

Let there be given $\frac{1}{1+xx}=\dot{z}$; by division it will be $\dot{z}=1-x^2+x^4-x^6$, &c. and thence (by PROB. II.) $z=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7$, &c. or else $\dot{z}=\frac{1}{x^2}-\frac{1}{x^4}+\frac{1}{x^6}$, and thence again by (PROB. II.) $z=-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}$, &c. $=$HDBH.

Let there be given $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}=\dot{z}$; by division it will be $\dot{z}=2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^2+34x^{\frac{5}{2}}$, and thence (by PROB. II.) $z=\frac{4}{3}x^{\frac{3}{2}}-x^2+\frac{14}{5}x^{\frac{5}{2}}-\frac{13}{3}x^3+\frac{68}{7}x^{\frac{7}{2}}$, &c.

EX. 4. *Where a previous Reduction is required by extraction of Roots.*

Let there be given $\dot{z}=\sqrt{aa+xx}$, (an Equation to the Hyperbola,) and the root being extracted to an infinite number of terms, it will be $\dot{z}=a+\frac{x^2}{2a}+\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{112a^7}$, whence, as in the foregoing, $z=ax+\frac{x^3}{6a}-\frac{x^5}{40a^3}+\frac{x^7}{112a^5}-\frac{5x^9}{1008a^7}$, &c.

In the same manner, if there were given $\dot{z}=\sqrt{aa-xx}$, (which is to the Circle,) there would be produced $z=ax-\frac{x^3}{6a}-\frac{x^5}{40a^3}-\frac{x^7}{112a^5}-\frac{5x^9}{1008a^7}$, &c.

And so if there were given $\dot{z}=\sqrt{x-xx}$, (an equation also to the Circle,) by extracting the root, there would arise $\dot{z}=x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{3}{2}}-\frac{1}{8}x^{\frac{5}{2}}-\frac{1}{16}x^{\frac{7}{2}}$, &c. and therefore $z=\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}-\frac{1}{72}x^{\frac{9}{2}}$, &c.

Thus $\dot{z}=\sqrt{aa+bx-xx}$, (an equation again to the Circle,) by extracting of the root, gives $\dot{z}=a+\frac{bx}{2a}-\frac{xx}{2a}-\frac{b^2x^2}{8a^3}$, &c. whence $z=ax+\frac{bx^2}{4a}-\frac{x^3}{6a}-\frac{b^2x^3}{24a^3}$, &c.

And

And thus $\sqrt{\frac{1+axx}{1-bxx}}=\dot{z}$, by a due reduction gives

$$\dot{z}=1\begin{array}{l}+\frac{1}{2}bx^2\\ \frac{1}{2}a\end{array}\begin{array}{l}+\frac{3}{8}bbx^4\\ +\frac{1}{4}ab\\ -\frac{1}{8}aa\end{array}, \text{ \&c. wence } z=x\begin{array}{l}+\frac{1}{6}bx^3\\ +\frac{1}{6}a\end{array}\begin{array}{l}+\frac{3}{40}bbx^5\\ +\frac{1}{20}ab\\ -\frac{1}{40}aa\end{array}, \text{ \&c.}$$

Thus finally, $\dot{z}=\sqrt[3]{a^3+x^3}$ by the extraction of the cubick root, gives $\dot{z}=a+\frac{x^3}{3a^2}-\frac{x^6}{9a^5}+\frac{5x^9}{81a^8}$, &c. and then by PROB. II. $z=ax+\frac{x^4}{12a^2}-\frac{x^7}{63a^5}+\frac{x^{10}}{162a^8}$, &c. $=$AFDB; or else $\dot{z}=x+\frac{a^3}{3xx}-\frac{a^6}{9x^5}+\frac{5a^9}{81x^8}$, &c. and thence $z=\frac{x^2}{2}-\frac{a^3}{3x}+\frac{a^6}{36x^4}-\frac{5a^9}{567x^7}$, &c. $=$HDBH.

Ex. 5. *Where a previous Reduction is required by the resolution of an affected equation.*

If a Curve be defined by this Equation $\dot{z}^3+a^2\dot{z}+ax\dot{z}-2a^3-x^3=0$; extract the root, and there will arise $\dot{z}=a-\frac{x}{4}+\frac{xx}{64a}+\frac{131x^3}{512aa}$, whence will be obtained as before $z=ax-\frac{xx}{8}+\frac{x^3}{192a}+\frac{131x^4}{2048a^2}$, &c.

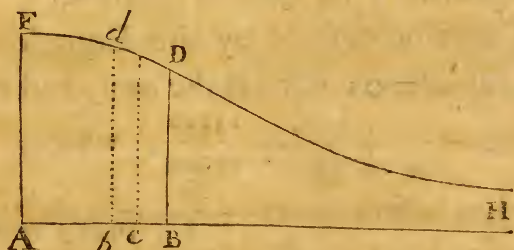
But if $\dot{z}^3-c\dot{z}^2-2x^2\dot{z}-c^2\dot{z}+2x^3+c^3=0$ were the equation to the Curve, the resolution will yield a Threefold Root; either $\dot{z}=c+x-\frac{xx}{4c}+\frac{x^3}{32c^2}$, &c. or $\dot{z}=c-x+\frac{3x^2}{4c}-\frac{15x^3}{32cc}$, &c. or $\dot{z}=-c-\frac{x^2}{2c}-\frac{x^3}{2cc}+\frac{x^5}{4c^4}$; and hence will arise the values of the Three corresponding Areas, $z=cx+\frac{1}{2}x^2-\frac{x^3}{12c}+\frac{x^4}{128c^2}$, &c. $z=cx-\frac{1}{2}x^2+\frac{x^3}{4c}-\frac{15x^4}{128c^2}$, &c. and $z=-cx-\frac{x^3}{6c}-\frac{x^4}{8c^2}+\frac{x^6}{24c^4}$, &c.

 I add

I add nothing concerning Mechanical Curves, becauſe their reduction to the form of Geometrical Curves will be taught afterwards.

But whereas the values of z thus found, belong to Areas, which are ſituate, ſometimes at a finite part, AB, of the Baſe or Abſciſs; ſometimes at a part BH, produced infinitely towards H; and ſometimes to both parts; according to their different terms: That the Due Value of the Area may be found, adjacent to any portion of the Abſciſs; That Area is always to be equal to the different values of z, which belong to the parts of the Abſciſs, that are terminated at the beginning and end of the Area.

AN INSTANCE: To the Curve expreſſed by the equation $\frac{1}{1+xx}=\dot{z}$, it is found that $z=x-\frac{1}{3}x^3+\frac{1}{5}x^5$, &c. Now that you may determine the



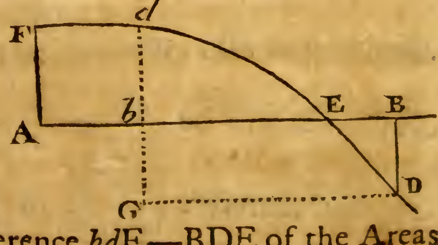
quantity of the Area bdDB adjacent to the part of the Baſe bB; from the value of z which ariſes by putting AB$=$X, take the value of z which ariſes by putting A$b=x$, (for diſtinction ſake writing X for AB and x for Ab,) and there is produced $X+\frac{1}{3}X^3+\frac{1}{5}X^5$, &c. $-x+\frac{1}{3}x^3+\frac{1}{5}x^5$, &c. $=bd$DB.

To the ſame Curve there is alſo found $z=-\frac{1}{x}+\frac{1}{3}x^3-\frac{1}{5}x^5$, &c. whence again, according to what is before obſerved, the Area $bd\text{DB}=\frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}$, &c. $-\frac{1}{X}+\frac{1}{3X^3}-\frac{1}{5X^5}$, &c. therefore if AB

or

or X be ſuppoſed infinite, the adjoining Area bdH towards H, which is alſo infinitely long, will be equivalent to $\frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5}$, *&c.* for the latter ſeries $-\frac{1}{X} + \frac{1}{3X^3} - \frac{1}{5X^5}$, *&c.* will vaniſh becauſe of its infinite denominators.

To the Curve repreſented by the equation $a + \frac{a^3}{xx} = \dot{z}$, it is found that $z = ax - \frac{a^3}{x}$, whence it is that $aX - \frac{a^3}{X} - ax + \frac{a^3}{x} =$ Area bdDB. To the Curve repreſented by the equation $a + \frac{a^3}{xx} = \dot{z}$, it is found that $z = ax - \frac{a^3}{x}$. Whence it is, that $aX - \frac{a^3}{X} - ax + \frac{a^3}{x} =$ Area bdDB; but this becomes infinite, whether x be ſuppoſed nothing, or X infinite; and therefore each Area, AFDB, and bdH, can be exhibited. And this always happens, when the Abſciſs x is found as well in the Numerators of ſome of the terms, as in the Denominators of others of the value of z. But when x is only found in the Numerators, as in the firſt example, the value of z belongs to the Area ſituate at AB on this ſide the Ordinate; and when it is only in the Denominators, as in the ſecond example, that Value, when the ſigns of all the terms are changed, belongs to the whole Area infinitely produced beyond the Ordinate.



If at any time the Curve Line cuts the Baſe or Abſciſs between the points b and B ſuppoſe in E, inſtead of the Area will be had the difference bdE—BDE of the Areas at at the different parts of the Base, to which if there be added the Rectangle BDGb, the Area dEDGb will be obtained.

But here it is chiefly to be regarded, that when in the value of $\dot{z}$, any term is divided by x of only one dimension, the Area corresponding to that term belongs to the *Conical* Hyperbola, and therefore may be exhibited by itself in an infinite series; as is done in what follows.

Let $\frac{a^3-a^2x}{ax+xx}=\dot{z}$ be an equation to a Curve; by division it becomes $\dot{z}=\frac{aa}{x}-2a+2x-\frac{2x^2}{a}+\frac{2x^3}{aa}$, &c. and thence $z=\left|\frac{aa}{x}\right|-2ax+x^2-\frac{2x^3}{3a}+\frac{x^4}{2a^2}$, &c. and the Area bdDB$=\left|\frac{aa}{X}\right|-2aX+X^2-\frac{2X^3}{3a}$, &c. $-\left|\frac{aa}{x}\right|+2ax-xx+\frac{2x^3}{3a}$, &c. Where by the marks, $\left|\frac{aa}{X}\right|$, and $\left|\frac{aa}{x}\right|$, I denote the little Areas belonging to the terms $\frac{aa}{X}$ and $\frac{aa}{x}$.

Now that $\left|\frac{aa}{X}\right|$ and $\left|\frac{aa}{x}\right|$ may be found; I make Ab or x to be definite, and bB indefinite or a Flowing Line, which therefore I call unity; so that it will be $\left|\frac{aa}{x+y}\right|$ equal to that Hyperbolical Area adjoining to bB; that is $\left|\frac{aa}{X}\right|-\left|\frac{aa}{x}\right|$. But by division it will be $\frac{aa}{x+y}=\frac{aa}{x}-\frac{a^2y}{x^2}+\frac{a^2y^2}{x^3}-\frac{a^2y^3}{x^4}$, &c. and therefore $\left|\frac{aa}{x+y}\right|$, or $\left|\frac{aa}{X}\right|-\left|\frac{aa}{x}\right|=\frac{a^2v}{x}$

$-a^2$

$-\frac{a^2y^2}{2x^2}+\frac{a^2y^3}{3x^3}-\frac{a^2y^4}{4x^4}$, &c. consequently the whole Area required bdDB $=\frac{a^2y}{x}-\frac{a^2y^2}{2x^2}+\frac{a^2y^3}{3x^3}$, &c. $-$ $2aX+X^2-\frac{2X^3}{3a}$, &c. $+2ax-x^2+\frac{2x^3}{3a}$, &c. After the same manner AB or X might have been used for a definite Line, and then it would have been $\left|\overline{\frac{aa}{X}}\right|-\left|\overline{\frac{aa}{x}}\right|=\frac{a^2y}{X}+\frac{a^2y^2}{2X^2}-\frac{a^2y^3}{3X^3}+\frac{a^2y^4}{4X^4}$, &c.

Moreover if bB be bisected in C, * and AC be assumed to be of a definite length, and Cb, CB indefinite; then making AC$=e$, and Cb or CB$=y$, it will be $bd=\frac{aa}{e-y}=\frac{aa}{e}+\frac{a^2y}{e^2}+\frac{a^2y^2}{e^3}+\frac{a^2y^3}{e^4}+\frac{a^2y^4}{e^5}$, &c. and therefore the Hyperbolick area adjacent to the part of the Abscis bC will be $\frac{a^2y}{e}+\frac{a^2y^2}{2e^2}+\frac{a^2y^3}{3e^3}+\frac{a^2y^4}{4e^4}$, &c. It will be also DB $=\frac{aa}{e+y}$ $=\frac{aa}{e}-\frac{aay}{e^2}+\frac{aay^2}{e^3}-\frac{aay^3}{e^4}+\frac{aay^4}{e^5}$, &c. and therefore the Area adjacent to the other part of the Abscis CB $=\frac{a^2y}{e}-\frac{a^2y^2}{2e^2}+\frac{a^2y^3}{3e^3}-\frac{a^2y^4}{4e^4}+\frac{a^2y^5}{5e^5}$, &c. and the sum of these Areas $\frac{2a^2y}{e}+\frac{2a^2y^3}{3e^3}+\frac{2a^2y^5}{5e^5}$, &c. will be equivalent to $\left|\overline{\frac{aa}{X}}\right|-\left|\overline{\frac{aa}{x}}\right|$.

* Fig. p. 124.

Thus in the equation $\dot{z}^3+\dot{z}^2+\dot{z}-x^3=0$ denoting the nature of a Curve, its root will be $\dot{z}=x$ $-\frac{1}{3}-\frac{2}{9x}+\frac{7}{81xx}+\frac{5}{81x^3}$, &c. whence there arises $z=\frac{1}{2}xx-\frac{1}{3}x-\left|\overline{\frac{2}{9x}}\right|-\frac{7}{81x}-\frac{5}{162xx}$, &c. and the

Area

Area $bdDB = \frac{1}{2}X^2 - \frac{1}{3}X - \left|\frac{2}{9X}\right| - \frac{7}{81X}$, &c. $-\frac{1}{2}xx + \frac{1}{3}x + \left|\frac{2}{9x}\right| + \frac{7}{81x}$, &c. *i. e.* $= \frac{1}{2}X^2 - \frac{1}{3}X - \frac{7}{81}X$, &c. $-\frac{1}{2}x^2 + \frac{1}{3}x + \frac{7}{81}x$, &c. $-\frac{4y}{9e} - \frac{4y^3}{27e^3} - \frac{4y^5}{45e^5}$, &c.

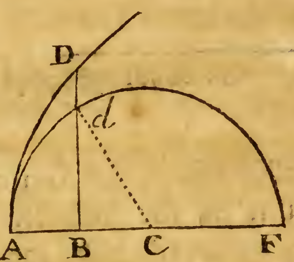
But this Hyperbolick term for the moſt part may be very commodiouſly avoided, by altering the beginning of the Abſciſs; that is, by increaſing or diminiſhing it by ſome given quantity. As in the former Example, where $\frac{a^3 - a^2x}{ax + xx} = \dot{z}$ was the equation to the Curve; if I would make b to be the beginning of the Abſciſs, ſuppoſing Ab to be of any determinate length, *viz.* $\frac{1}{2}a$, for the remainder of the Abſciſs bB, I ſhall now write x: ſo that, if I diminiſh the Abſciſs by $\frac{1}{2}a$, by writing $x + \frac{1}{2}a$ inſtead of x, it will become $\frac{\frac{1}{2}a^3 - a^2x}{\frac{3}{4}a^2 + 2ax + x^2} = \dot{z}$; and by diviſion $\dot{z} = \frac{2}{3}a - \frac{28}{9}x + \frac{200x^2}{27a}$, &c. whence ariſes $z = \frac{2}{3}ax - \frac{14}{9}x^2 + \frac{200x^3}{81a}$, &c. $= bd$DB.

Alſo the equation $\frac{a^3 - a^2x}{ax + xx} = \dot{z}$ might have been reſolved into the Two infinite ſeries, $\dot{z} = \frac{a^3}{x^2} - \frac{a^4}{x^3} + \frac{a^5}{x^4}$, &c. $-a + x - \frac{xx}{a} + \frac{x^3}{a^2}$, &c. where there is found no term divided by the firſt power of x.

But ſuch kind of ſeries, where the powers of x aſcend infinitely in the numerators of one, and in the denominators of the other, are not ſo proper to derive the value of z from by Arithmetical Computation, when the ſpecies are to be changed into Numbers.

Scarce any difficulty can occur to any one, who is to undertake ſuch a computation in Numbers, after the value of the Area is obtained in ſpecies. Yet for the more compleat illuſtration of the foregoing doctrine, I ſhall add an Example or Two.

Let the Hyperbola AD be propoſed, whoſe equation is $\sqrt{x+xx}=z$, its vertex being at A, and each of its Axes equal to unity; from what goes before, its Area ADB$=\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}+\frac{1}{72}x^{\frac{9}{2}}-\frac{5}{704}x^{\frac{11}{2}}$, *&c.* that is, $x^{\frac{1}{2}}$ into $\frac{2}{3}x+\frac{1}{5}x^{2}-\frac{1}{28}x^{3}+\frac{1}{72}x^{4}-\frac{5}{704}x^{5}$, *&c.* which ſeries may be infinitely produced by multiplying the laſt term continually by the ſucceeding terms of this progreſſion, $\frac{1.3}{2.5}x$. $\frac{-1.5}{4.7}x$. $\frac{-3.7}{6.9}x$. $\frac{-5.9}{8.11}x$. $\frac{-7.11}{10.13}x$, *&c.* that is, the



firſt term $\frac{2}{3}x^{\frac{3}{2}}$ multiplied by $\frac{1.3}{2.5}x$, makes the ſecond term $\frac{1}{5}x^{\frac{5}{2}}$; which multiplied by $\frac{-1.5}{4.7}x$, makes the third term $\frac{-1}{28}x^{\frac{7}{2}}$; which multiplied by $\frac{-3.7}{6.9}$, makes the fourth term $+\frac{1}{72}x^{\frac{9}{2}}$. And ſo on *ad infinitum*.

Now let AB be aſſumed of any length, ſuppoſe $\frac{1}{4}$, and writing this number for x, and its root $\frac{1}{2}$ for $x^{\frac{1}{2}}$, the firſt term $\frac{2}{3}x^{\frac{3}{2}}$ or $\frac{2}{3}\times\frac{1}{8}$ being reduced to a decimal fraction, becomes 0.083333333, *&c.* this into $\frac{1.3}{2.5.4}$ makes 0 . 00625 the ſecond term; this into $\frac{-1.5}{4.7.4}$ makes 0 . 0002790178, *&c.* the third term. And ſo on for ever. But the terms thus reduced by degrees, I dispose into Two Tables; the affirmative terms in One, and the Negative in Another, and add them up as you see here.

+ 0 . 0833333333333333	— 0 . 0002790178571429
6250000000000	34679066051
271267361111	834465027
5135169396	26285354
144628917	961296
4954581	38676
190948	1663
7963	75
352	4
16	0 . 0002825719389575
1	+ 0 . 0896109885646618
0 . 0896109885646618	0 . 0893284166257043

Then from the sum of the affirmative, I take the sum of the negative terms, and there remains 0.0893284166257043 for the quantity of the Hyperbolick Area A*d*B which was to be found.

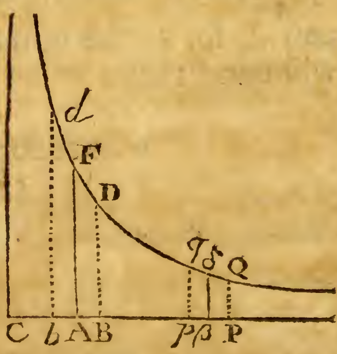
Let the Circle A*d*F [*See the same Fig.*] be proposed, which is expressed by the equation $\sqrt{x-xx}=z$, whose diameter is unity; and from what goes before its Area A*d*B will be $\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}-\frac{1}{72}x^{\frac{9}{2}}$, &c. in which series, since the terms do not differ from the terms of the series which above expressed the Hyperbolick Area, except in the signs + and —; nothing else remains to be done, than to connect the same numeral terms with their signs; that is, by subtracting the connected sums of both the forementioned Tables, 0.0898935605036193, from the first term doubled 0.1666666666666666, &c. and the remainder 0.0767731061630473 will be the portion A*d*B of the Circular Area, supposing AB to be a fourth part

part of the *Diameter*. And hence we may observe, that though the Areas of the Circle and Hyperbola are not expressed in a Geometrical consideration, yet each of them is discovered by the same Arithmetical computation.

The portion of the Circle A*d*B being found, from thence the whole Area may be derived. For the radius *d*C being drawn, multiply B*d* or $\frac{1}{4}\sqrt{3}$ into BC or $\frac{1}{4}$, and one half of the product $\frac{1}{32}\sqrt{3}$, or 0.0541265877365275 will be the value of the Triangle C*d*B; which added to the Area *Ad*B, will give the Sector AC*d*, 0.1308996938995747; the Sextuple of which 0.7853981633974482 is the whole Area.

And hence (by the way) the length of the Circumference will be 3.14159265358979 28, which is found by dividing the Area by a fourth part of the diameter.

To this we shall add the calculation of the Area comprehended between the Hyperbola *d*FD and its Asymptote CA, let C be the center of the Hyperbola, and putting CA $=a$, AF $=b$, and AB $=$ Ab $=x$; it will be $\frac{ab}{a+x}=$ BD, and $\frac{ab}{a-x}=bd$; whence the Area AFDB $=bx-\frac{bxx}{2a}+\frac{bx^3}{3a^2}-\frac{bx^4}{4a^3}$, &c. And the



Area AF$db=bx+\frac{bx^2}{2a}+\frac{bx^3}{3a^2}+\frac{bx^4}{4a^3}$, &c. And the sum bdDB $=2bx+\frac{2bx^3}{3a^2}+\frac{2bx^5}{5a^4}+\frac{2bx^7}{7a^6}$, &c. Now let us suppose CA $=$ AF $=1$, and Ab or AB $=\frac{1}{10}$, Cb being $=0.9$, and CB $=1.1$. then substituting

ing these numbers for a, b, and x, the first term of the series becomes 0 . 2, the second 0 . 000666666666666, &c. the third 0 . 000004, and so on; as you see in this table.

$$\begin{array}{r} 0 . 2000000000000000 \\ 6666666666666 \\ 40000000000 \\ 285714286 \\ 2222222 \\ 18182 \\ 154 \\ 1 \\ \hline 0 . 2006706954621511 \end{array} = \text{Area } bd\text{DB}.$$

If the parts of this Area Ad and AD be added separately, subtract the lesser DA from the greater dA, and there will remain $\frac{bx^2}{a} + \frac{bx^4}{2a^3} + \frac{bx^6}{3a^5} + \frac{bx^8}{4a^7}$, &c. where, if 1 be wrote for a and b, and $\frac{1}{10}$ for x, the terms being reduced to decimals will stand thus.

$$\begin{array}{r} 0 . 0100000000000000 \\ 500000000000 \\ 3333333333 \\ 25000000 \\ 200000 \\ 1667 \\ 14 \\ \hline 0 . 0100503358535014 \end{array} = \text{A}d - \text{AD}.$$

Now if this difference of the Areas be added to, and subtracted from, their sum before found; half the aggregate 0 . 1053605156578263 will be the greater

greater Area A*d*; and half the remainder 0.0953101798043248 will be the leffer Area AD.

By the fame tables thefe Areas AD and A*d* will be obtained alfo, when AB and A*b* are fuppofed $\frac{1}{100}$, or CB=1.01, and C*b*=0.99; if the numbers are but duly transferred to lower places. As

0.0200000000000000	0.0001000000000000
6666666666	50000000
4000000	3333
28	
Sum 0.0200006667066695 = *b*D	0.0001000050003333 = A*d*—AD

Half the aggregate 0.0100503358535014=A*d* and Half the refidue 0.0099503308531681=AD.

And fo putting AB and A*b*=$\frac{1}{1000}$, or CB=1.001, and C*b*=0.999, there will be obtained A*d*=0.00100050003335835 and AD=0.00099950013330835.

In the fame manner (if CA and AF=1) putting AB and A*b*=0.2, or 0.02, or 0.002, thefe areas will arife.

A*d*=0.2231435513142097 and AD=0.1823215576939546
or A*d*=0.0202027073175194 and AD=0.0198026272961797
or A*d*=0.002002 and AD=0.001

From thefe Areas thus found it will be eafy to derive others by addition and fubtraction alone, for as it is $\frac{1.2}{0.8} \times \frac{1.2}{0.9}=2$; the fum of the areas 0.6931471805599453 belonging to the ratios $\frac{1.2}{0.8}$ and $\frac{1.2}{0.9}$ (that is infifting upon the parts of the abfcifs 1.2, 0.8. and 1.2, 0.9.) will be the area AF$\delta\beta$, when Cβ =2; as is known. Again, fince $\frac{1.2}{0.8}$; × 2=3, the fum 1.0986122886681097 of of the areas belonging to $\frac{1.2}{0.8}$ and 2, will be the area AF$\delta\beta$; when Cβ = 3. Again, as it is $\frac{2\times 2}{0.8}=5$; and $2\times 5=10$; by a due addition of Areas will be obtained 1.6093379124341004 = AF$\delta\beta$, when Cβ = 5: and 2.302585092994 0457 = AF$\delta\beta$, when Cβ = 10. And since $10\times 10=100$; and $10\times 100=1000$; and $\sqrt{5\times 10\times 0.98}=7$; and $10\times 1.1=11$; and $\frac{1000\times 1.001}{7\times 11}=13$; and $\frac{1000\times 0.998}{2}=499$; it is plain that the area AF$\delta\beta$ may be found by the composition of the areas found before, when Cβ = 100; 1000; 7; or any other of the above-mentioned numbers: CA = AF being still unity.

Thus I was willing to insinuate, that a method might be derived from hence, very proper for the construction of a canon of Logarithms, which determines the Hyperbolical Areas, (from which the Logarithms may easily be derived,) corresponding to so many prime numbers, as it were by Two operations only; which are not very troublesome. But whereas that Canon seems to be derivable from this fountain more commodiously than from any other, what if I should point out its construction here to compleat the whole?

First therefore, having assumed 0 *for the Logarithm of the number* 1; *and* 1 *for the Logarithm of the number* 10, *as is generally done; the Logarithms of the Prime numbers* 2, 3, 5, 7, 11, 13, 17, 37, *are to be investigated by dividing the Hyperbolical Areas now found by* 2.302585092994 0457, *which is the Area corresponding to the number* 10; *or, which is the same thing, by multiplying by its reciprocal* 0.43429 448190 32518. *Thus for instance, if* 0.69314718, *&c. the Area corresponding to the number* 2, *were multiplied by* 0.43429, *&c. it makes* 0.30102999 5663981 2 *the Logarithm of the number* 2.

Then

Then the Logarithms of all the numbers in the canon, which are made by the multiplication of this, are to be found by the addition of their Logarithms as is usual; and the void places are to be filled up afterwards by the help of this Theorem.

Let N *be a number to which a Logarithm is to be adapted;* x *the difference between that and the Two nearest numbers equally distant on each side, whose Logarithms are already found; and let* d *be half the difference of their Logarithms; then the required Logarithm of the number* N *will be obtained by adding* $d+\frac{dx}{2N}+\frac{dx^3}{12N^3}$, *&c. to the Logarithm of the lesser number. For if the numbers are expounded by* $C\pi$, $C\beta$, *and* CP; *the rectangle* CBD *or* $C\beta\delta=1$ *as before, and the ordinates* PQ *and* pq *being raised; if* N *be written for* $C\beta$, *and* x *for* βp, *or* βP, *the area* $pqQP$ *or* $\frac{2x}{N}+\frac{2x^3}{3N^3}+\frac{2x^5}{5N^5}$, *&c. will be to the Area* $pq\delta\beta$ *or* $\frac{x}{N}+\frac{x^2}{2N^2}+\frac{x^3}{3N^3}$, *&c. as the difference between the Logarithms of the extreme numbers or* $2d$, *is to the difference between the Logarithms of the lesser, and of the middle one; which therefore will be* $\dfrac{\frac{dx}{N}+\frac{dx^2}{2N^2}+\frac{dx^3}{3N^3},\ \&c.}{\frac{x}{N}+\frac{x^3}{3N^3}+\frac{x^5}{5N^5},\ \&c.}$; *that is, when the division is performed* $d+\frac{dx}{2N}+\frac{dx^3}{12N^3}$, *&c.*

The two first terms of this series $d+\frac{dx}{2N}$, *I think to be accurate enough for the construction of a canon of Logarithms, even though they were to be produced to fourteen or fifteen figures; provided the number, whose Logarithm is to be found, be less than* 1000; *and this can give little trouble in the calculation, because* x *is generally an unit, or the number* 2. *Yet it is not necessary to interpolate all the places by the help of this rule; for the Logarithms of numbers, which are produced by* *the multiplication or division of the number last found, may be obtained by the numbers whose Logarithms were had before by the addition or subtraction of their Logarithms. Moreover by the differences of their Logarithms, and by their second and third differences, (if there be occasion,) the void places may more expeditiously be supplied; the foregoing Rule being to be applied only, where the continuation of some full places is wanted, in order to obtain these differences.*

By the same method Rules may be found for the intercalation of Logarithms, when of three numbers, the Logarithms of the lesser and of the middle number are given, or of the middle number and of the greater; and this altho' the numbers should not be in Arithmetical progression.

Also by pursuing the steps of this method, Rules might be easily discovered for the construction of the tables of Artifical Sines and Tangents, without the assistance of the Natural Tables.——But these things only by the bye.

Hitherto we have treated only of the quadrature of Curves, which are expressed by equations, consisting of complicate terms; and that by means of their reduction to equations, which consist of an infinite number of simple terms. But whereas such Curves may sometimes be squared by finite equations also; or however may be compared with other Curves, whose areas in a manner may be considered as known, of which kind are the Conick Sections. For this reason I thought fit to adjoin the two following Catalogues or Tables of Theorems according to my promise, constructed by the help of the Seventh and Eighth Propositions aforegoing.

The first of these exhibits the Areas of such Curves as can be squared; and the Latter contains such Curves whose areas may be compared with the

Areas

Areas of the Conick Sections. In each of these, the *Latin* Letters, *d*, *e*, *f*, *g*, and *h*, denote any given Quantities; x and z the Bases or Abscisses of Curves; v and y Parallel Ordinates; and s and t Areas, as before. The *Greek* Letters η and θ annexed to the quantity z, denote the number of the dimensions of the said z, whether it be Integer, or Fraction, Affirmative, or Negative. As if $\eta=3$, then $z^{\eta}=z^{3}$, $z^{2\eta}=z^{6}$ $z^{-\eta}=z^{-3}$ or $\frac{1}{z^{3}}$, $z^{\eta+1}=z^{4}$, and $z^{\eta-1}=z^{2}$.

Moreover in the values of the Areas, for the sake of brevity, is written R instead of these Radicals $\sqrt{e+fz^{\eta}}$ or $\sqrt{e+fz^{2\eta}}$, by which the value of the Ordinate y is affected.

See Table the First.

T Other

A TABLE *of some Curves related to Rectilinear Figures, constructed by* PROBLEM VII.

		Order of Curves.	Values of the Areas.
I		$dz^{n-1}=y$	$\frac{d}{n}z^{n}=t$
II		$\frac{dz^{n-1}}{ee+2efz^{n}+ffz^{2n}}=y$	$\frac{dz^{n}}{ne^{2}+nefz^{n}}=t$
III	1	$dz^{n-1}\sqrt{e+fz^{n}}=y$	$\frac{2d}{3nf}R^{3}=t$
	2	$dz^{2n-1}\sqrt{e+fz^{n}}=y$	$\frac{-4e+6fz^{n}}{15nff}dR^{3}=t$
	3	$dz^{3n-1}\sqrt{e+fz^{n}}=y$	$\frac{16ee-24efz^{n}+30ffz^{2n}}{105nf^{3}}dR^{3}=t$
	4	$dz^{4n-1}\sqrt{e+fz^{n}}=y$	$\frac{-96e^{3}+144e^{2}fz^{n}-180ef^{2}z^{2n}+210f^{3}z^{3n}}{945nf^{4}}dR^{3}=t$
IV	1	$\frac{dz^{n-1}}{\sqrt{e+fz^{n}}}=y$	$\frac{2d}{nf}R=t$
	2	$\frac{dz^{2n-1}}{\sqrt{e+fz^{n}}}=y$	$\frac{-4e+2fz^{n}}{3nff}dR=t$
	3	$\frac{dz^{3n-1}}{\sqrt{e+fz^{n}}}=y$	$\frac{16e^{2}-8efz^{n}+6ffz^{2n}}{15nf^{3}}dR=t$
	4	$\frac{dz^{4n-1}}{\sqrt{e+fz^{n}}}=y$	$\frac{-96e^{3}+48e^{2}fz^{n}-36ef^{2}z^{2n}+30f^{3}z^{3n}}{105nf^{4}}dR=t$

pag. 137.

Catalogue you have the proposed Curve represented by the line QEχR, the beginning of whose Abscis is A, the Abscis AC, the Ordinate CE, the beginning of the Area $\alpha\chi$, and the area described $\alpha\chi$EC. But the beginning of this Area or the Initial Term (which commonly either commen- of the Abscis A, or recedes ce) is found by seeking the Aα, where the value of the by erecting the Perpendicu-

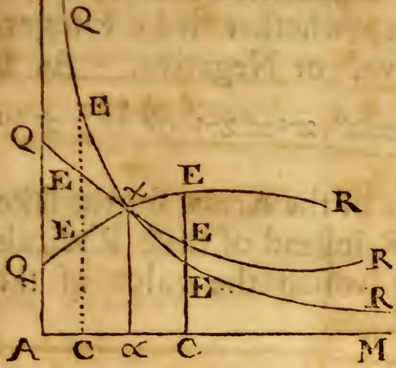
anner you have the Conick

the line PDG, aGD or GDS whose

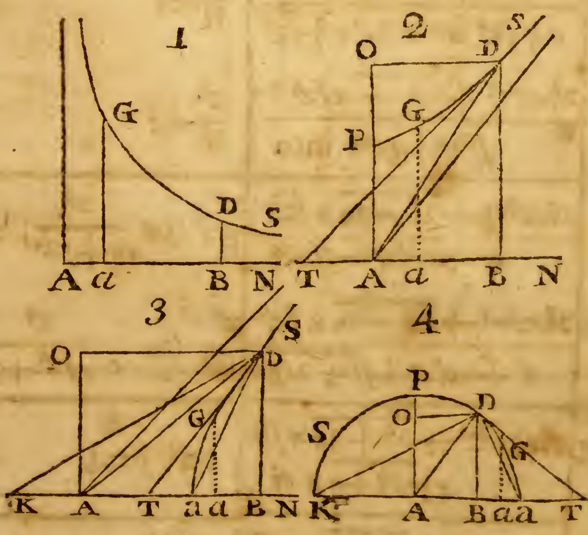
To these may be added the following more general Theorems, by which a way is prepared to others of a higher Order. Let p be here put for $\sqrt{b+iz^{\eta}}$.

		Order of Curves.	Values of the Areas.
V	1	$2\theta e z^{\theta-1} \begin{matrix}+2\theta f\\ +3\eta f\end{matrix} z^{\theta+\eta-1}$ into $\frac{1}{2}\sqrt{e+fz^{\eta}}=y$	$z^{\theta}R^{3}=t$
	2	$2\theta e z^{\theta-1} \begin{matrix}+2\theta f\\ +3\eta f\end{matrix} z^{\theta+\eta-1} \begin{matrix}+2\theta g\\ +6\eta g\end{matrix} z^{\theta+2\eta-1}$ into $\frac{1}{2}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z^{\theta}R^{3}=t$
VI	1	$\dfrac{2\theta e z^{\theta-1}+\overline{2\theta+\eta}\times fz^{\theta+\eta-1}}{2\sqrt{e+fz^{\eta}}}=y$	$z^{\theta}R=t$
	2	$\dfrac{2\theta e z^{\theta-1}+\overline{2\theta+\eta}\times fz^{\theta+\eta-1}+\overline{2\theta+2\eta}\times gz^{\theta+2\eta-1}}{2\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\theta}R=t$
VII	1	$\dfrac{2\theta e z^{\theta-1}+\overline{2\theta-\eta}\times fz^{\theta+\eta-1}}{e+fz^{\eta} \text{ into } 2\sqrt{e+fz^{\eta}}}=y$	$\dfrac{z^{\theta}}{R}=t$
	2	$\dfrac{2\theta\eta z^{\theta-1}+\overline{2\theta-\eta}\times fz^{\theta+\eta-1}+\overline{2\theta-2\eta}\times gz^{\theta+2\eta-1}}{ffz^{\eta}+gz^{2\eta} \text{ into } 2\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$\dfrac{z^{\theta}}{R}=t$
VIII	1	$\dfrac{2\theta e z^{\theta-1}+\overline{2\theta-2\eta}\times fz^{\theta+\eta-1}}{ee+2efz^{\eta}-ffz^{2\eta}}=2y$	$\dfrac{z^{\theta}}{R^{2}}$ (or $\dfrac{z^{\theta}}{e+fz^{\eta}}$) $=t$
	2	$\dfrac{2\theta e z^{\theta-1}+\overline{2\theta-2\eta}\times fz^{\theta+\eta-1}+\overline{2\theta-4\eta}\times gz^{\theta+2\eta-1}}{e^{2}+2efz^{\eta}+\overline{ff+2eg}\times z^{2\eta}+2fgz^{3\eta}+ggz^{4\eta}}=2y$	$\dfrac{z^{\theta}}{R^{2}}$ (or $\dfrac{z^{\theta}}{e+fz^{\eta}+gz^{2\eta}}$) $=t$
IX		$2\theta ebz^{\theta-1}+\overline{2\theta+3\eta}\times fbz^{\theta+\eta-1}+\overline{2\theta+4\eta}\times fiz^{\theta+2\eta-1}$ $+\overline{2\theta+\eta}\times ei$ into $\dfrac{\sqrt{e+fz^{\eta}}}{2\sqrt{b+iz^{\eta}}}=y$	$z^{\theta}R^{3}p=t$
X		$2\theta ebz^{\theta-1}+\overline{2\theta+3\eta}\times fbz^{\theta+\eta-1}+\overline{2\theta+2\eta}\times fiz^{\theta-2\eta-1}$ $+\overline{2\theta-\eta}\times ei$ into $\dfrac{\sqrt{e+fz^{\eta}}}{b+iz^{\eta} \text{ into } 2\sqrt{b+iz^{\eta}}}=y$	$\dfrac{z^{\theta}R^{3}}{p}=t$

Other things of the same kind might be added. But I shall now pass unto another sort of Curves which may be compared with the *Conick Sections.* And in this Table or Catalogue you have the proposed Curve represented by the line QEχR, the beginning of whose Abscis is A, the Abscis AC, the Ordinate CE, the beginning of the Area $\alpha\chi$, and the area described $\alpha\chi$EC. But the beginning of this Area or the Initial Term (which commonly either commences at the beginning of the Abscis A, or recedes to an infinite distance) is found by seeking the length of the Abscis Aα, where the value of the Area is nothing; and by erecting the Perpendicular $\alpha\chi$.



After the same manner you have the Conick



Section represented by the line PDG, aGD or GDS whose

whose center is A; vertex a; rectangular semidiameters Aa and AP; the beginning of the Abscifs A, or *a*, or *α*; the Abscifs AB, or aB, or *a*B; the Ordinate BD; the Tangent DT, meeting AB in T; the Subtense AD; and the Rectangle inscribed and adscribed ABDO.

Therefore retaining the Letters before defined, it will be AC$=z$, CE$=y$, $\alpha\chi$EC$=t$, AB or *a*B or aB$=x$, BD$=v$, and ABDP or *a*GDB$=s$. And besides when two Conick Sections are required for the determination of any Area, the Area of the latter shall be called σ, the Abscifs ξ, and the Ordinate Υ.

See Table the Second.

Before I go on to illustrate by examples the Theorems that are delivered in this Catalogue of Curves, I think it may be proper to premise the following observations.

1. Whereas in the Equations representing Curves, I have all along supposed all the signs of the quantities d, e, f, g, h, and i, to be affirmative; whensoever it shall happen that they are negative, they must be changed in the subsequent values of the Abscis and Ordinate of the *Conick Section*; and also of the Area required.

2. Also the signs of the numeral symbols η and θ, when they are negative, must be changed in the values of the Areas. Moreover their signs being changed, the Theorems themselves may acquire a new form. Thus in the fourth form of the latter Table, the sign of η being changed, the Third Theorem becomes $\frac{d}{z^{-2\eta+1}\sqrt{e+fz^{-\eta}}}=y$, $\frac{1}{z^{-\eta}}=x$, &c. that is, $\frac{dz^{3\eta-1}}{\sqrt{ez^{2\eta}+fz^{\eta}}}=y$, $z^{\eta}=x$, $\sqrt{fx+ex^2}=v$, $\frac{d}{\eta e}$ into $2xv-3s=t$. And the same may be observed in others.

3. But in the second Table, the series of the First, Second, Third, Fourth, Ninth and Tenth Orders, are produced, *in infinitum*, by division alone. Thus having $\frac{dz^{4\eta-1}}{e+fz^{\eta}}=y$; if you perform the division to a convenient period, there will arise $\frac{d}{f}z^{3\eta-1}-\frac{de}{ff}z^{2\eta-1}+\frac{de^2}{f^3}z^{\eta-1}-\frac{\frac{de^3}{f^3}z^{\eta-1}}{e+fz^{\eta}}=y$: The three first terms belong to the fourth order of the first Table; and the fourth belongs to the first species of this order. Whence it appears that the Area

A TABLE *of some Curves related to the Conick Sections, constructed by* PROBLEM VIII.

		Forms of Curves.	Conick Sections. Absciss.	Conick Sections. Ordinate.	Areas of the Curves.
I	1	$\frac{dz^{n-1}}{e+fz^n}=y$	$z^n=x$	$\frac{d}{e+fx}=v$	$\frac{1}{\eta}s=t=\frac{aGDB}{\eta}$. *Fig.* 1.
	2	$\frac{dz^{2n-1}}{e+fz^n}=y$	$z^n=x$	$\frac{d}{e+fx}=v$	$\frac{d}{\eta f}z^n-\frac{e}{\eta f}s=t.$
	3	$\frac{dz^{3n-1}}{e+fz^n}=y$	$z^n=x$	$\frac{d}{e+fx}=v$	$\frac{d}{2\eta f}z^{2n}-\frac{de}{\eta f^2}z^n+\frac{e^2}{\eta f^2}s=t.$
II	1	$\frac{dz^{\frac{1}{2}n-1}}{e+fz^n}=y$	$\sqrt{\frac{d}{e+fz^n}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$	$\frac{2xv-4s}{\eta}=t=\frac{4}{\eta}ADGa.$ *Fig.* 3. 4.
	2	$\frac{dz^{\frac{3}{2}n-1}}{e+fz^n}=y$	$\sqrt{\frac{d}{e+fz^n}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$	$\frac{2d}{\eta f}z^{\frac{1}{2}n}+\frac{4es-2exv}{\eta f}=t.$
	3	$\frac{dz^{\frac{5}{2}n-1}}{e+fz^n}=y$	$\sqrt{\frac{d}{e+fz^n}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$	$\frac{2d}{3\eta f}z^{\frac{3}{2}n}-\frac{2de}{\eta f^2}z^{\frac{1}{2}n}+\frac{2e^2xv-4e^2s}{\eta f^2}=t.$
III	1	$\frac{d}{z}\sqrt{e+fz^n}=y$	$\frac{1}{z^n}=x^2$	$\sqrt{f+ex^2}=v$	$\frac{4de}{\eta f}\times\overline{\frac{v^3}{2ex}-s}=t=\frac{4de}{\eta f}$ into aGDT, or into APDB—TDB. *Fig.* 2. 3. 4.
		or thus	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{8de^2}{\eta f^2}\times\overline{s-\frac{1}{2}xv-\frac{fv}{4e}+\frac{f^2v}{4e^2x}}=t=\frac{8de^2}{\eta f^2}$ into aGDA $+\frac{f^2v}{4e^2x}$. *Fig.* 3. 4.
	2	$\frac{d}{z^{n+1}}\sqrt{e+fz^n}=y$	$\frac{1}{z^n}=x^2$	$\sqrt{f+ex^2}=v$	$\frac{-2d}{\eta}s=t=\frac{2d}{\eta}$ APDB or $\frac{2d}{\eta}$ aGDB. *Fig.* 2. 3. 4.
		or thus	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{4de}{\eta f}\times\overline{s-\frac{1}{2}xv-\frac{fv}{2e}}=t=\frac{4de}{\eta f}\times$ aGDK. *Fig.* 3. 4.
	3	$\frac{d}{z^{2n+1}}\sqrt{e+fz^n}=y$	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{-d}{\eta}s=t=\frac{d}{\eta}\times$ —aGDB or BDPK. *Fig.* 4.
	4	$\frac{d}{z^{3n+1}}\sqrt{e+fz^n}=y$	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{3dfs-2dv^3}{6\eta e}=t.$

4. Some of these orders may also otherwise be derived from others. As in the Last Table, the Fifth, Sixth, Seventh, and Eleventh from the Eighth; and the Ninth from the Tenth. So that I might have omitted them, but that they may be of some use, though not altogether necessary. Yet I have omitted some Orders, which I might have derived from the First and Second; and also from the Ninth and Tenth; because they were affected by denominators that were complicate, and therefore can hardly be of any use.

5. If the defining equation of any Curve be compounded of several equations of different orders, or of different species of the same order; the Area must be compounded of the corresponding Areas. Take care however that they be rightly connected with their proper signs; for we must not always add or subtract at the same time Ordinates to or from Ordinates, or corresponding Areas to or from corresponding Areas; but sometimes the sum of these, and the difference of those, is to be taken, for a new Ordinate; or to constitute a corresponding Area: And this must be done, when the constituent Areas are posited on the contrary side of the Ordinate. But that the cautious Geometrician may the more readily avoid these inconveniencies, I have prefixed their proper signs to the several values of the Areas, though sometimes negative. As is done in the Fifth and Seventh Orders of the Last Table.

6. It is farther to be observed about the signs of the Areas; that $+s$ denotes, either that the Area of the Conick Section adjoining to the Abscis is to be added to the other quantities in the value of t; [*See the first Example following*;] or that the Area on the other side of the Ordinate is to be subtracted. And on the contrary $-s$ denotes, ambiguously, either that the Area adjacent to the Abscis is

		Conick Sections. Absciss.	Conick Sections. Ordinate.	Areas of the Curves.
VI 1	$\frac{dz^{\frac{1}{2}n-1}}{e+fz+gz^{2n}}=y$	$\sqrt{\frac{2dg}{f-p+2gz^n}}=x$; $\sqrt{\frac{2dg}{f+p+2gz^n}}=\xi$	$\sqrt{d+\frac{-f+p}{2g}x^2}=v$; $\sqrt{d+\frac{-f-p}{2g}\xi^2}=\Upsilon$	$\frac{2xv-4s-2\xi\Upsilon+4\sigma}{np}=t.$
VI 2	$\frac{dz^{\frac{3}{2}n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{2dgz^n}{fz^n-pz^n+2e}}=x$; $\sqrt{\frac{2dgz^n}{fz^n+pz^n+2e}}=\xi$	$\sqrt{d+\frac{-f+p}{2e}x^2}=v$; $\sqrt{d+\frac{-f-p}{2e}\xi^2}=\Upsilon$	$\frac{4s-2xv-4\sigma+\xi\Upsilon}{np}=t.$
VII 1	$\frac{d}{z}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$; $\frac{1}{z^n}=\xi$	$\sqrt{e+fx+gx^2}=v$; $\sqrt{e+f\xi+g\xi^2}=\Upsilon$	$\frac{4de^2\xi\Upsilon+2def\Upsilon-2dfgxv{+4deg \atop -2ffd}v-8de^2\sigma+4dfgs}{4neg-nff}=t.$
VII 2	$dz^{n-1}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{d}{n}s=t=\frac{d}{n}\,a$GDB. *Fig.* 2. 3. 4.
VII 3	$dz^{2n-1}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{d}{3ng}v^3-\frac{df}{2ng}s=t.$
VII 4	$dz^{3n-1}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{6dgx-5df}{24ng^2}v^3+\frac{5df^2-4deg}{16ng^2}s=t.$
VIII 1	$\frac{dz^{n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{8dgs-4dgxv-2dfv}{4neg-nf^2}=t=\frac{8dg}{4neg-nf^2}\times a\mathrm{GDB}\mp\triangle\mathrm{DBA}$. *Fig.* 2. 4.
VIII 2	$\frac{dz^{2n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{-4dfs+2dfxv+4dev}{4neg-nf^2}=t.$
VIII 3	$\frac{dz^{3n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{{3dff \atop -4deg}s\,{-2dff \atop +4deg}xv-2defv}{4neg^2-nf^2g}=t.$
VIII 4	$\frac{dz^{4n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{{36defg \atop -15df^3}s\,{+8degg \atop -2df^2g}x^2v\,{+10df^3 \atop -28defg}xv\,{+10deff \atop -16de^2g}v}{24neg^3-6nf^2g^2}=t.$

		Forms of Curves.	Conick Sections. Abscifs.	Conick Sections. Ordinate.	Areas of the Curves.
IV	1	$\frac{d}{z\sqrt{e+fz^n}}=y$	$\frac{1}{z^n}=x^2$	$\sqrt{f+ex^2}=v$	$\frac{4d}{\eta f}\times\overline{\frac{1}{2}xv-s}=t=\frac{4d}{\eta f}$ into PAD or into aGDA. *Fig.* 2. 3. 4.
		or thus	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{8de}{\eta f^2}\times\overline{s-\frac{1}{2}xv-\frac{fv}{4e}}=t=\frac{8de}{\eta f^2}$ into aGDA. *Fig.* 3. 4.
	2	$\frac{d}{z^{n+1}\sqrt{e+fz^n}}=y$	$\frac{1}{z^n}=x^2$	$\sqrt{f+ex^2}=v$	$\frac{2d}{\eta e}\times\overline{s-xv}=t=\frac{2d}{\eta e}$ into POD or into AODGa. *Fig* . 2. 3. 4.
		or thus	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{4d}{\eta f}\times\overline{\frac{1}{2}xv-s}=t=\frac{4d}{\eta f}$ into aDGa. *Fig.* 3. 4.
	3	$\frac{d}{z^{2n+1}\sqrt{e+fz^n}}=y$	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{d}{\eta e}\times\overline{3s-2xv}=t=\frac{d}{\eta e}$ into 3aGDa—ΔaDB. *Fig.* 3. 4.
	4	$\frac{d}{z^{3n+1}\sqrt{e+fz^n}}=y$	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{10dfxv-15dfs-2dex^2v}{6\eta e^2}=t.$
V	1	$\frac{dz^{n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\sqrt{\frac{d}{g}+\frac{f^2-4eg}{4g^2}x^2}=v$	$\frac{xv-2s}{\eta}=t.$
		or thus	$\sqrt{\frac{dz^{2n}}{e+fz^n+gz^{2n}}}=x$	$\sqrt{\frac{d}{e}+\frac{f^2-4eg}{4e^2}x^2}=v$	$\frac{2s-xv}{\eta}=t.$
	2	$\frac{dz^{2n-1}}{e+fz^n+gz^{2n}}=y$	$\left\{\begin{array}{l}\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x\\ fz^n+gz^{2n}=\xi\end{array}\right.$	$\left.\begin{array}{l}\sqrt{\frac{d}{g}+\frac{f^2-4eg}{4g^2}x^2}=v\\ \frac{1}{e+\xi}=\Upsilon\end{array}\right\}$	$\frac{d\sigma+2fs-fxv}{2\eta g}=t.$

Area is $\frac{d}{3\eta f}z^{3\eta} - \frac{de}{2\eta ff}z^{2\eta} + \frac{de^2}{\eta f^3}z^{\eta} - \frac{e^3}{ef^3}s$; putting s for the Area of the *Conick Section*, whose Abscis $x = z^{\eta}$, and Ordinate $v = \frac{d}{e + fx}$.

The series of the Fifth and Sixth Orders may be infinitely continued by the help of the two Theorems in the fifth order of the First Table, by a due addition or subtraction; as also the series of the Seventh and Eighth, by the help of the Theorems in the following sixth Order. And the series of the eleventh, by the help of the Theorems in the Tenth Order of the same First Table. For instance, if the series of the said Fifth Order were to be continued; suppose $\theta = -4\eta$, and the first Theorem of the Fifth Order of the other Table will become $-8\eta ez^{-4\eta-1} - 5\eta fz^{-3\eta-1}$ into $\frac{1}{2}\sqrt{e + fz^{\eta}} = y$; $\frac{R^3}{z^{4\eta}} = t$: but according to the fourth Theorem of this series to be produced, writing $-\frac{5\eta f}{2}$ for d, it is $-\frac{5\eta}{2}fz^{-3\eta-1}\sqrt{e + fz^{\eta}} = y$, $\frac{1}{z^{\eta}} = x$, $\sqrt{fx + exx} = v$, and $\frac{10fv^3 - 15f^2s}{12e} = t$; so that subtracting the former values of y and t, there will remain $4\eta ez^{-4\eta-1}\sqrt{e + fz^{\eta}} = y$, and $\frac{10fv^3 - 15f^2s}{12e} - \frac{R^3}{z^{4\eta}} = t$; these being multiplied by $\frac{d}{4\eta e}$, and (if you please) for $\frac{R^3}{z^{4\eta}}$ writing xv^3, there will arise a Fifth Theorem of the series to be produced, $\frac{d}{z^{4\eta-1}}\sqrt{e + fz^{\eta}} = y$, $\frac{1}{z^{\eta}} = x$, $\sqrt{fx + exx} = v$, and $\frac{10dfv^3 - 15df^2s}{48\eta e^2} - \frac{dxv^3}{4\eta e} = t$.

4. Some

		Forms of Curves.	Conick Sections. Abscis.	Conick Sections. Ordinate.	Areas of the Curves.
IX	1	$\frac{dz^{\eta-1}\sqrt{e+fz^{\eta}}}{g+hz^{\eta}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$	$\frac{\begin{matrix}4fg\\-4eh\end{matrix}s\begin{matrix}-2fg\\+2eh\end{matrix}xv+2df\frac{v}{x}}{\eta fh}=t.$
IX	2	$\frac{dz^{2\eta-1}\sqrt{e+fz^{\eta}}}{g+hz^{\eta}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$	$\frac{\begin{matrix}4egh\\-4fg^2\end{matrix}s\begin{matrix}-2egh\\+2fgg\end{matrix}xv+\frac{2}{3}dh\frac{v^3}{x^3}-2dfg\frac{v}{x}}{\eta fh^2}=t.$
X	1	$\frac{dz^{\eta-1}}{g+hz^{\eta}\sqrt{e+fz^{\eta}}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$	$\frac{2xv-4s}{\eta f}=t=\frac{4}{\eta f}$ ADGa. *Fig.* 3. 4.
X	2	$\frac{dz^{2\eta-1}}{g+hz^{\eta}\sqrt{e+fz^{\eta}}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$	$\frac{4gs-2gxv+2d\frac{v}{x}}{\eta fh}=t.$
XI	1	$dz^{-1}\sqrt{\frac{e+fz^{\eta}}{g+hz^{\eta}}}=y$	$\sqrt{g+hz^{\eta}}=x$ $\sqrt{h+gz^{\eta}}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}=v$ $\sqrt{\frac{fg-eh}{g}+\frac{e}{g}\xi^2}=\Upsilon$	$\frac{2dxv^3z^{-\eta}-4dfs-4de\sigma}{\eta fg-\eta eh}=t.$
XI	2	$dz^{\eta-1}\sqrt{\frac{e+fz^{\eta}}{g+hz^{\eta}}}=y$	$\sqrt{g+hz^{\eta}}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}=v$	$\frac{2d}{\eta h}s=t.$
XI	3	$dz^{2\eta-1}\sqrt{\frac{e+fz^{\eta}}{g+hz^{\eta}}}=y$	$\sqrt{g+hz^{\eta}}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}=v$	$\frac{dhxv^3\begin{matrix}-3dfg\\-deh\end{matrix}s}{2\eta fh^2}=t.$

is to be ſubtracted; or that the Area on the other ſide of the Ordinate is to be added; as it may ſeem convenient. Alſo the value of t, if it comes out affirmative, denotes the Area of the Curve propoſed adjoining to its Abſciſs; and contrariwiſe, if it be negative, it repreſents the Area on the other ſide of the Ordinate.

7. But that this Area may more certainly be defined, we muſt enquire after its limits. And as to its limit at the Baſe or Abſciſs, at the Ordinate, and at the Perimeter of the Curve, there can be no uncertainty. But its initial limit, or the beginning, from whence its deſcription commences, may obtain various poſitions. In the following Examples, it is either at the beginning of the Abſciſs; or at an infinite diſtance; or in the concourſe of the Curve with its Abſciſs. But it may be placed elſewhere: And wherever it is, it may be found by ſeeking that length of the Abſciſs, at which the value of t becomes nothing, and there erecting an Ordinate; for the Ordinate ſo raiſed will be the limit required.

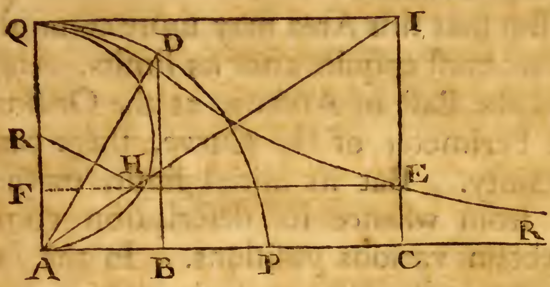
8. If any part of the Area be poſited below the Abſciſs, t will denote the difference of that, and of the part above the Abſciſs.

9. Whenſoever the dimenſions of the terms in the values of x, v, and t, ſhall aſcend too high, or deſcend too low; they may be reduced to a juſt degree by dividing or multiplying by any given quantity (which may be ſuppoſed to perform the office of unity,) ſo often, as the dimenſions ſhall be either too high or too low.

10. Beſides the foregoing Catalogues or Tables, we might alſo conſtruct Tables of Curves related to other Curves, which may be the moſt ſimple in their kind. As $\sqrt{e+fx^3}=v$; or $x\sqrt{e+fx^3}=v$; or $\sqrt{e+fx^4}=v$, &c. ſo that we might at all times derive the Area of any propoſed Curve from the ſimpleſt

ſimpleſt original; and know to what Curves it ſtands related. But now let us illuſtrate by examples what has been already delivered.

EXAMPLE I. Let QER be a Conchoidal of ſuch a kind that the ſemicircle QHA being de-



ſcribed, and AC, being erected perpendicular to the diameter AQ, if the parallelogram QACI be completed, the diagonal AI be drawn meeting the ſemicircle in H, and from H the perpendicular HE be let fall to IC; then the point E will deſcribe a Curve, whoſe area ACEQ is ſought.

Therefore make $AQ=a$, $AC=z$, $CE=y$; and becauſe of the continual proportionals AI, AQ, AH, EC; it will be EC or $y=\frac{a^3}{a^2+z^2}$.

Now that this may acquire the form of the equations in the Tables, make $\eta=2$, and for z^2 in the denominator write z^{η}; and $a^3z^{\frac{1}{2}\eta-1}$ for a^3 or a^3z^{1-1} in the numerator; and there will ariſe $y=\frac{a^3z^{\frac{1}{2}\eta-1}}{a^2+z^{\eta}}$; an equation of the firſt ſpecies of the ſecond order of the laſt Table: and the terms being compared, it will be $d=a^3$, $e=a^2$, and $f=1$. So that $\sqrt{\frac{a^3}{a^2+z^2}}=x$, $\sqrt{a^3-a^2x^2}=v$, and $xv-2s=t$.

Now

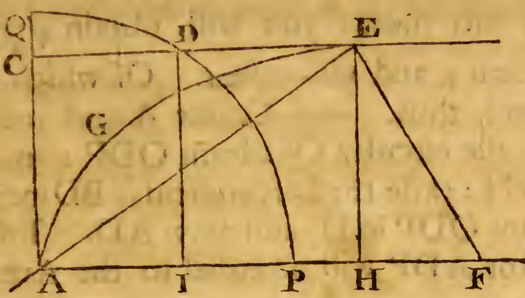
Now that the values found of x and v may be reduced to a juſt number of dimenſions, chuſe any given quantity as a, by which, as unity, a^3 may be multiplied once in the value of x; and in the value of v, a^3 may be divided once, and a^2x^2 twice; and by this means you will obtain $\sqrt{\frac{a^4}{a^2+z^2}}=x$; $\sqrt{a^2-x^2}=v$; and $xv-2s=t$. Of which the conſtruction is thus. ——Center A and radius AQ deſcribe the circular Quadrant QDP; in AC take AB=AH; raiſe the Perpendicular BD meeting the Quadrant QDP in D, and draw AD. Then twice the Sector ADP will be equal to the Area ſought ACEQ. For $\sqrt{\frac{a^4}{a^2+z^2}}=(\sqrt{AD^2-AB^2}=)$ BD or v; and $xv-2s=2\triangle$ADB$-$2ABDQ, or $=2\triangle$ ADB+2BDP; that is, either $=-2$QAD, or $=2$DAP. Of which values the affirmative 2DAP belongs to the Area ACEQ on this ſide EC; and the negative -2QAD belongs to the Area RECR extended *ad infinitum* beyond EC.

The ſolution of Problems thus found may ſometimes be made more elegant. Thus in the preſent caſe, drawing RH the ſemidiameter of the Circle QHA, becauſe of the Equal arches QH and DP, the Sector QRH is half the Sector DAP; and therefore a fourth part of the ſurface ACEQ.

Ex. 2. Let AGE be a Curve which is deſcribed by the angular point E of the *Norma* AEF, whilſt one of the legs AE, being indeterminate, paſſes continually through the given point A; and the other CE of a given length ſlides upon the right line AF given in poſition. Let fall EH perpendicular to AF, and complete the parallelogram AHEC; then calling AC$=z$, CE$=y$, and EF $=a$, becauſe of HF, HE, HA, continual proportionals, it will be HA or $y=\frac{z^2}{\sqrt{a^2-z^2}}$.

U Now

Now that the Area AGEC may be known, ſuppoſe $z^2 = z^n$, or $2 = n$, and thence it will be $\frac{z^{\frac{3}{2}n-1}}{\sqrt{a^2-z^n}} = y$. Here ſince z in the numerator is of a

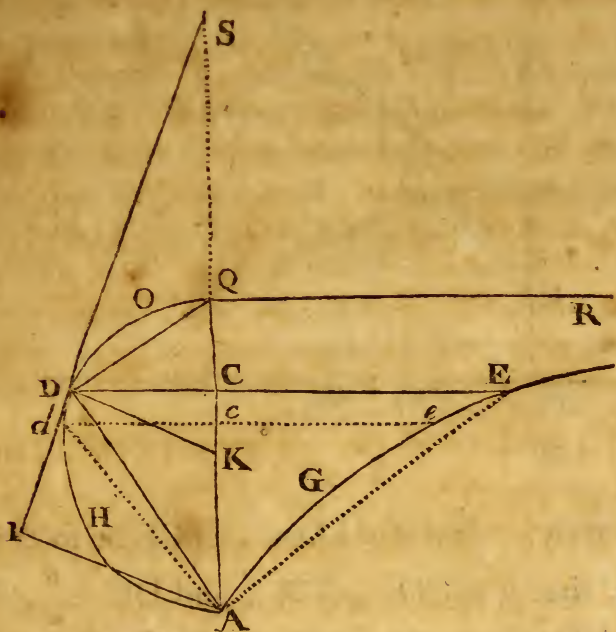


fracted dimenſion, depreſs the value of y by dividing by $z^{\frac{1}{2}n}$, and it will be $\frac{z^{n-1}}{\sqrt{a^2 z^{-n}-1}} = y$; an Equation of the ſecond ſpecies of the ſeventh Order of the latter Table; and the terms being compared it is $d = 1$, $e = -1$, and $f = a^2$. So that $z^2 = (\frac{1}{z^{-n}} =)\ x^2$, $\sqrt{a^2-x^2} = v$, and $s - xv = t$. Therefore ſince x and z are equal, and ſince $\sqrt{a^2-x^2} = v$ is an equation to a circle, whoſe diameter is a; Center A, diſtance a, or EF, let the Circle PDQ be deſcribed, which CE meets in D; and let the parallelogram ACDI be completed. Then will AC $= z$, CD $= v$, and the Area ſought AGEC $= s - xv =$ ACDP $-$ ACDI $=$ IDP.

Ex. 3. Let AGE be the Ciſſoid belonging to the Circle ADQ, deſcribed with the diameter AQ. Let DCE be drawn perpendicular to the diameter, and meeting the Curves in D and E. Then naming AC $= z$, CE $= y$, and AQ $= a$; becauſe of CD, CA, CE continual proportionals, it will be CE or $y = \frac{zz}{\sqrt{az-zz}}$; and dividing by z, it is $y = \frac{z}{\sqrt{az^{-1}-1}}$.

Therefore

Therefore $z^{-1}=z^{n}$, or $-1=n$, and thence $y=\frac{z^{-2n-1}}{\sqrt{az^{n}-1}}$, an Equation of the third ſpecies of the

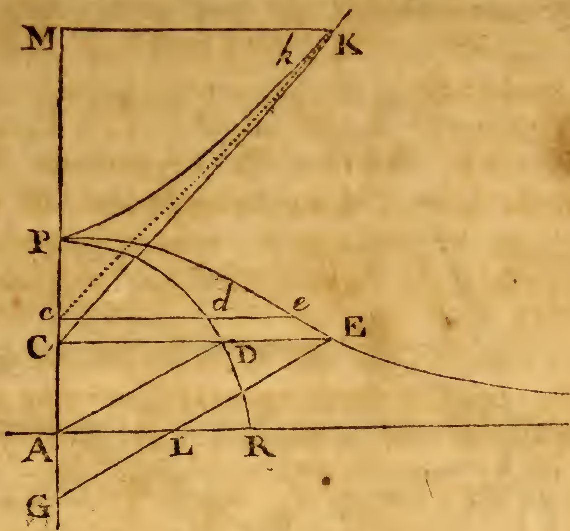


fourth Order of the ſecond Table. The terms being compared it is $d=1$, $e=-1$, and $f=a$. Therefore $z=\frac{1}{z^{n}}=x$, $\sqrt{ax-xx}=v$, and $3s-2xv=t$. Whence it is AC$=x$, CD$=v$; and thence ACDH$=s$: So that 3ACDH$-4\triangle$ADC$=3s-2xv=t=$ Area of the Ciſſoid ACEGA; or, which is the ſame thing, 3 Segment ADHA$=$ Area ADEGA, or 4 Segment ADHA$=$ Area AHDEGA.

Ex. 4. Let PE be the firſt Conchoid of the Antients deſcribed from Center G with the aſymptote AL, and diſtance LE; draw its axis GAP, and let fall the ordinate EC. Then call AC$=z$, CE$=y$, GA$=b$, and AP$=c$, becauſe of the proportionals AC : CE$-$AL : : GC : CE, it will be CE or $y=\frac{b+z}{z}\sqrt{c^{2}-z^{2}}$.

 Now

Now that the area PEC may be found from hence, the parts of the Ordinate CE are to be considered



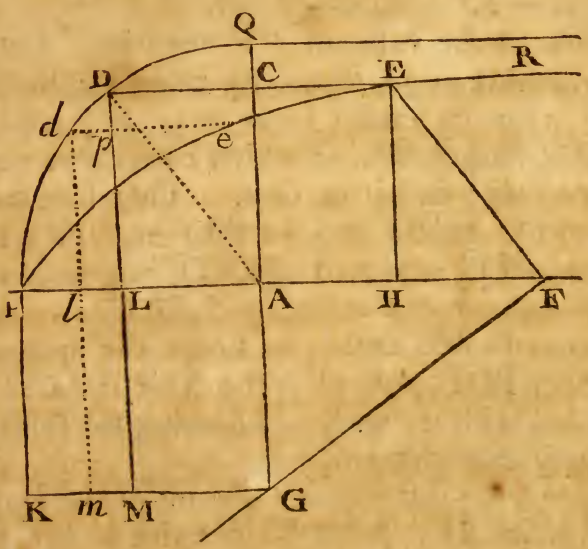
separately. And if the Ordinate CE be so divided in D, that it is $CD = \sqrt{e^2 - z^2}$, and $DE = \frac{b}{z}\sqrt{e^2 - z^2}$, then CD will be the Ordinate of a Circle described from center A, and with radius AP. Therefore the part of the Area PDC is known, and there will remain the other part DPED to be found. But since DE, that part of the Ordinate by which it is described is equivalent to $\frac{b}{z}\sqrt{e^2 - z^2}$; suppose $2 = \eta$, and it becomes $\frac{b}{z}\sqrt{e^2 - z^\eta} = DE$, an Equation of the first species of the third Order of the second Table. The terms therefore being compared, it is, $d = b$, $e = c^2$, and $f = -1$; and therefore $\frac{1}{z} = \frac{1}{\sqrt{z^\eta}} = x$, $\sqrt{-1 + c^2x^2} = v$, and $2bc^2s - \frac{bv^3}{x} = t$.

These things being found; reduce them to a just number of dimensions, by multiplying the terms that are too depressed, and dividing those that are too

too high, by some given quantity. If this be done by c, there will arise $\frac{c^2}{z} = x$, $\sqrt{-c^2 + x^2} = v$, and $\frac{2bs}{c} - \frac{bv^3}{cx} = t$: The construction of which is in this manner.

With center A, principal Vertex P, and parameter 2AP, describe the Hyperbola PK; and from the point C draw the right line CK that shall touch the Parabola in K: then it will be as AP to 2AG, so is the Area CKPC to the Area required DPED.

Ex. 5. Let the *Norma* GFE revolve about the Pole G, so that its angular point F may continu-



ally slide upon the right line AF given in position; then conceive the Curve PE to be described by any point E in the other leg EF. Now that the Area of this Curve may be found, let fall GA and EH perpendicular to the right line AF, and completing the parallelogram AHEC, call AC$=z$, CE$=y$, AG $=b$, and EF$=c$; then because of the proportionals HF

HF : EH : : AG : AF, we ſhall have AF $= \frac{bz}{\sqrt{cc-zz}}$. Therefore CE or $y = \frac{bz}{\sqrt{c^2-z^2}} - \sqrt{c^2-z^2}$; but whereas $\sqrt{c^2-z^2}$ is the ordinate of a Circle, deſcribed with the ſemidiameter c, about the center A; let ſuch a Circle PDQ be deſcribed, which CE produced meets in D; then it will be DE$= \frac{bz}{\sqrt{cc-zz}}$; by the help of which Equation, there remains the Area PDEP or DERQ to be determined. Suppoſe $\eta = 2$ and $\theta = b$, and it will be DE$= \frac{bz^{\eta-1}}{\sqrt{cc-z^\eta}}$; an Equation of the firſt ſpecies of the fourth Order of the firſt Table: and the terms being compared it will be, $b = d$, $cc = e$, and $-1 = f$. So that $-b\sqrt{cc-zz} = -b\text{R} = t$.

Now as the value of t is negative, and therefore the Area repreſented by it lies beyond the line DE; that its initial limit may be found, ſeek for that length of z at which t becomes nothing, and you will find it to be c. Therefore continue AC to Q, that it may be AQ$=c$, and erect the Ordinate QR; then DQRED will be the Area whoſe value now found is $-b\sqrt{cc-zz}$.

If you ſhould deſire to know the quantity of the Area PDE, poſited at the Abſciſs AC, and coextended with it, without knowing the limit QR; you may thus determine it.

From the value which t obtains at the length of the Abſciſs AC, ſubtract its value at the beginning of the Abſciſs; that is, from $-b\sqrt{cc-zz}$ ſubtract $-bc$, and there will ariſe the deſired quantity $bc - b\sqrt{cc-zz}$. Therefore complete the parallelogram PAGK, and let fall DM perpendicular to AP, which meets GK in M, and the parallelogram PKML will be equal to the Area PDE.

When-

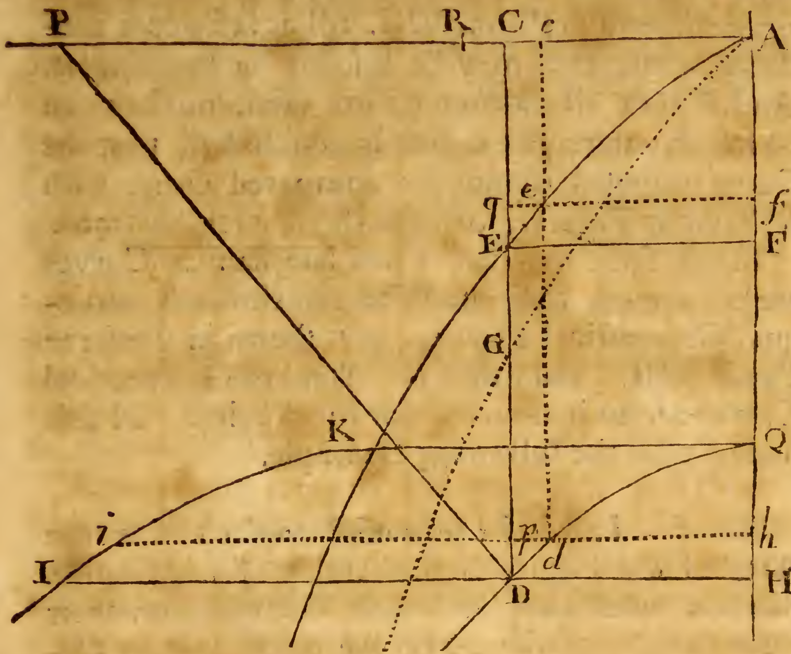
Whenever the Equation defining the nature of the Curve cannot be found in the Tables, nor can be reduced to simpler terms by division, nor by any other means; it must be transformed into other Equations of Curves related to it after the manner shewn in PROB. VIII. till at last one is produced whose area may be known by the Tables. And if after all endeavours are used, no such can be found; it may be certainly concluded, that the Curve proposed cannot be compared either with Rectilinear Figures, or with the Conick Sections.

In the same manner, when Mechanical Curves are concerned, they must be transformed into equal Geometrical Figures, as is shewn in the same PROB. VIII. And then the Areas of Geometrical Curves are to be found from the Tables. Of this matter take the following Example.

Ex. 6. Let it be proposed to determine the Area of the Figure of the Arches of any Conick Section, when they are made Ordinates on their right sines. As let A be the center of the Conick Section; AQ, AR, the Semi-axes; CD the Ordinate to the Axis AR; and PD a perpendicular at the point D. Also let AE be the said mechanical Curve meeting CD in E. From its nature before defined CE will be equal to the arch QD; therefore the Area AEC is sought, or completing the parallelogram ACEF, the Excess AEF is required. To which purpose let a be the *Latus Rectum* of the Conick Section, and b its *Latus Transversum* or 2AQ. Also let AC $=z$ and CD$=y$: then it will be $\sqrt{\frac{1}{4}bb+\frac{b}{a}zz}=y$, an equation to a Conick Section as is known. Also PC$=\frac{b}{a}z$; and thence PD$=\sqrt{\frac{1}{4}bb+\frac{bb+ab}{aa}zz}$.

Now

Now ſince the Fluxion of the Arch QD is to the Fluxion of the Abſciſs AC, as PD to CD; if the Fluxion of the Abſciſs be ſuppoſed 1, the Fluxion of the Arch QD, or of the Ordinate CE, will be



$\sqrt{\dfrac{\frac{1}{4}bb+\frac{bb+ab}{aa}zz}{\frac{1}{4}bb+\frac{b}{a}zz}}$; draw this into FE or $\dot{z}$, and there will ariſe $\dot{z}\sqrt{\dfrac{\frac{1}{4}bb+\frac{bb+ab}{aa}zz}{\frac{1}{4}bb+\frac{b}{a}zz}}$ for the Fluxion of the Area AEF. If therefore in the Ordinate CD you take $CG=z\sqrt{\dfrac{\frac{1}{4}bb+\frac{bb+ab}{aa}zz}{\frac{1}{4}bb+\frac{b}{a}zz}}$, then the Area AGC, which is deſcribed by CG moving upon AC, will be equal to the Area AEF; and the Curve

Curve AG will be a Geometrical Curve; therefore the Area AGC is sought. To this purpose, let z^n be substituted for z^2 in the last equation and

it becomes $z^{n-1}\sqrt{\dfrac{\frac{1}{4}bb+\frac{bb+ab}{aa}z^n}{\frac{1}{4}bb+\frac{b}{a}z^n}}=CG$; an Equation of the second species of the eleventh Order of the second Table. And from a comparison of terms it is $d=1$, $e=\frac{1}{4}bb=g$, $f=\dfrac{bb+ab}{aa}$, $h=\dfrac{b}{a}$: So that $\sqrt{\frac{1}{4}bb+\frac{b}{a}zz}=x$, $\sqrt{-\dfrac{b^3}{4a}+\dfrac{a+b}{a}xx}=v$, and $\dfrac{b}{a}s=t$: that is, $CD=x$, $DP=v$, and $\dfrac{a}{v}s=t$. And this is the Construction of what is now found.

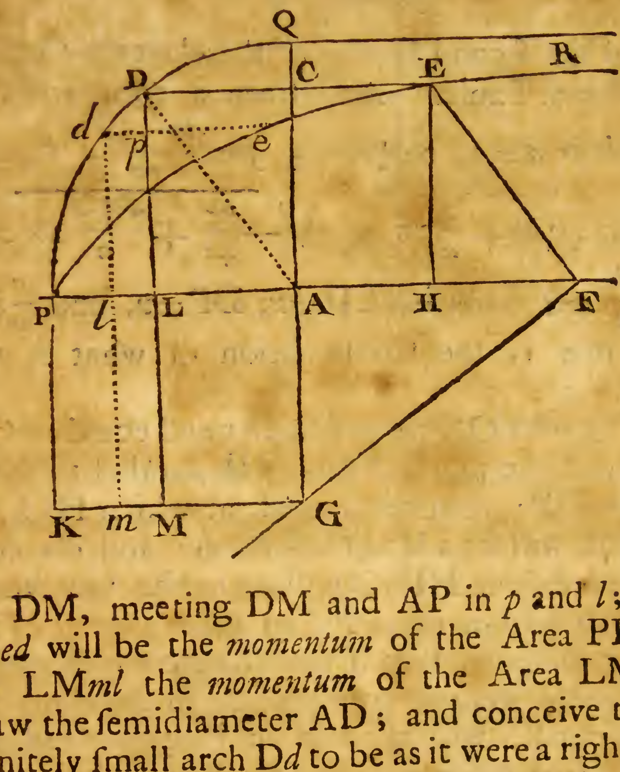
At Q erect QK perpendicular and equal to QA, and thro' the point D draw HI parallel to it but equal to DP; then the line KI, at which HI is terminated, will be a Conick Section; and the comprehended Area HIKQ will be to the Area sought AEF, as b to a, or as PC to AC.

Here observe, that if you change the sign of b, the Conick Section, to whose Arch the right line CE is equal, will become an Ellipsis; and besides, if you make $b=-a$, the Ellipsis becomes a Circle. And in this case the Line KA becomes a Right Line parallel to AQ.

After the Area of any Curve has been thus found and constructed, we should consider about the demonstration of the construction, that laying aside all Algebraical Calculations as much as may be, the Theorem may be adorned, and made elegant, so as to become fit for publick view. Now there is a general method of demonstrating, which I shall endeavour to illustrate by the following examples.

Demonstration of the Construction in Example 5.

In the Arch PQ take a point *d* indefinitely near to D, and draw *de* and *dm* parallel to DE



and DM, meeting DM and AP in *p* and *l*; then DE*ed* will be the *momentum* of the Area PDEP, and LM*ml* the *momentum* of the Area LMKP. Draw the semidiameter AD; and conceive the indefinitely small arch D*d* to be as it were a right line, and the Triangles D*pd*, ALD, will be like, and therefore D*p* : *pd* : : AL : LD. But it is HF : EH : : AG : AF; that is, AL : LD : : ML : DE. And therefore D*p* :*pd* : : ML : DE. Wherefore D*p* × DE=*pd* × ML; that is, the *momentum* DE*ed* is equal to the *momentum* LM*ml*. And since this is demonstrated indeterminately of any contemporaneous moments whatever, it is plain that all the moments of the Area PDEP are equal to all the contemporaneous moments of the Area PLMK, and therefore the whole Areas composed of these moments are equal to each other. Q. E. D.

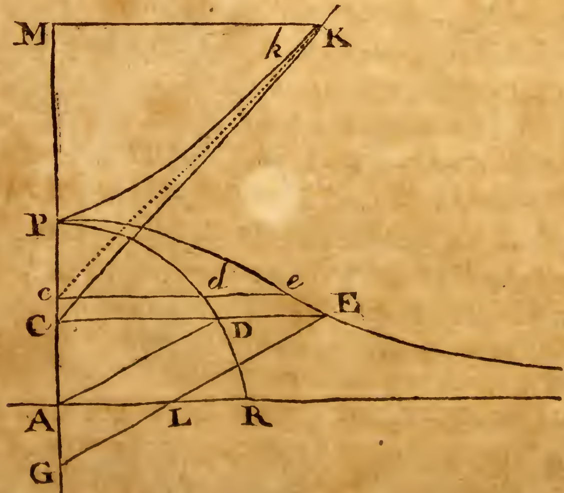
Demonstration

Demonstration of the Construction in Example 3.

Let DE*ed* [*Fig. p.* 147.] be the *moment* of the superficies AHDE, and A*d*DA the contemporary moment of the segment ADH. Draw the semidiameter DK, and let *de* meet AK in *c*. Then it is C*c* : D*d* : : CD : DK. Besides it is DC : QA (2DK) : : AC : DE ; and therefore C*c* : 2D*d* : : DC : 2DK : : AC : DE, and C*c* × DE = 2D*d* × AC. Now to the moment of the periphery D*d* produced, that is, to the Tangent of the Circle, let fall the perpendicular AI, and AI will be equal to AC ; so that 2D*d* × AC = 2D*d* × AI = 4△*s*AD*d*, so that 4△*s*AD*d* = C*c* × DE = the moment DE*ed*. Therefore every moment of the space AHDE is quadruple of the contemporary moment of the segment ADH, and consequently that whole Space is quadruple of the whole Segment Q. E. D.

Demonstration of the Construction in Example 4.

Draw *ce* parallel to CE and at an indefinitely small distance from it, and the Tangent of the

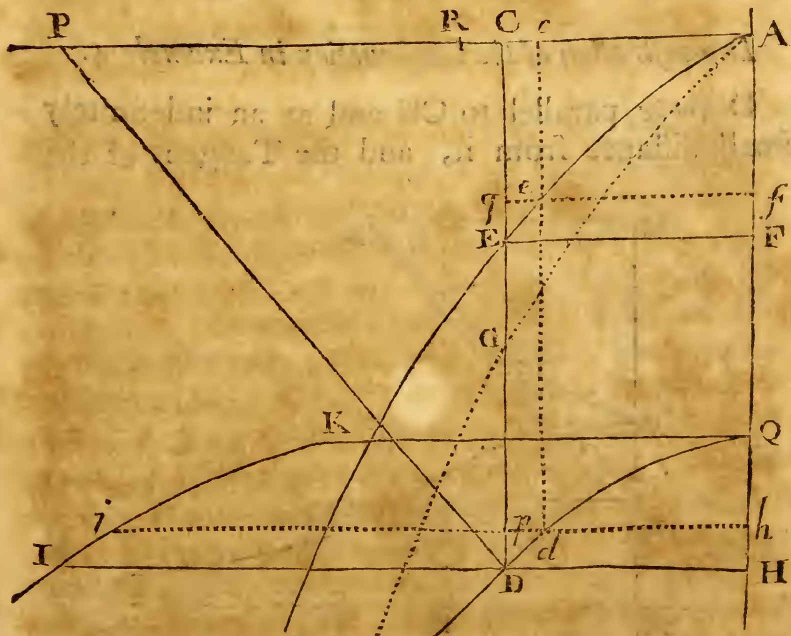


Hyperbola *ck* ; and let fall KM perpendicular to

AP. Now from the nature of the Hyperbola, it will be AC : AP : : AP : AM. And therefore $\overline{AG}^2 : \overline{GL}^2 :: \overline{AC}^2 : \overline{LE}^2$ (or $\overline{AP}^2$) $:: \overline{AP}^2 : \overline{AM}^2$; and *divisim* $\overline{AG}^2 : \overline{AL}^2 (\overline{DE}^2) :: \overline{AP}^2 : \overline{AM}^2 - \overline{AP}^2 (\overline{MK}^2)$; and *inverse* AG : AP : : DE : MK. But the little Area DE*ed* is to the Triangle CK*c*, as the altitude DE is to $\frac{1}{2}$ the altitude KM; that is, as AG : $\frac{1}{2}$AP; wherefore all the moments of the space PDE are to all the comporary moments of the space PKC, as AG : $\frac{1}{2}$AP; and consequently those whole Spaces are the in the same ratio. Q. E. D.

Demonstration of the Construction in Example 6.

Draw *cd* parallel and infinitely near to CD meeting the Curve AE in *e*, and draw *hi* and *fe* meet-



ing DC in *p* and *q*; then by the Hypothesis D*d* =E*q*, and from the similitude of the Triangles D*dp* and DCP, it will will be D*p* : (D*d*) E*q* : : CP : (PD)

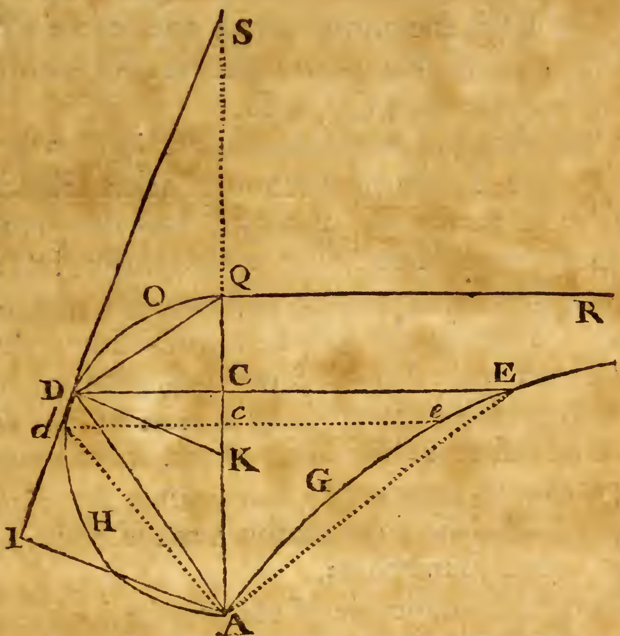
: (PD) HI; so that $Dp \times HI = Eq \times CP$; thence $Dp \times HI$ (the moment $HIib$) : $Eq \times AC$ (the moment $EFfe$) : $Eq \times CP$: $Eq \times AC$: : CP : AC. Wherefore since PC and AC are in the given ratio of the *Latus Transversum* to the *Latus Rectum* of the Conick Section QD; and since $HIib$ and $EFfe$ the moments of the Areas HIKQ and AEF are in that ratio; the Areas themselves will be in same ratio Q. E. D.

In this kind of demonstrations, it is to be observed, that I assume such Quantities for equal, whose ratio is that of equality: and that is to be esteemed a ratio of equality, which differs less from equality than by any unequal ratio that can be assigned. Thus in the last demonstration I supposed the rectangle $Eq \times AC$ or $FEqf$, to be equal to the space $FEef$ (because by reason of the difference Eqe infinitely less than them, or nothing in comparison of them, they have not a ratio of inequality. For the same reason I made $DP \times HI = HIib$. And so in others.

I have here made use of this method of proving Areas of Curves to be equal, or to have a given ratio, by the equality, or by the given ratio of their moments, because it has an affinity to the usual methods in these matters. But that seems more natural, which depends upon the generation of Superficies by Motion or Fluxion. Thus if the Construction in the second Example was to be demonstrated: From the nature of the Circle, the Fluxion of the right line ID [*Fig. p.* 146.] is to the Fluxion of the right line IP, as AI to ID; and it is AI : ID : : ID : CE from the nature of the Curve AGE; and therefore $CE \times \dot{ID} = IDDG \times \dot{IP}$. But $CE \times \dot{ID}$ is equal to the Fluxion of the Area PDI; and therefore those Areas being generated by equal Fluxions must be Equal. Q. E. D.

For

For the ſake of farther illuſtration, I ſhall add the demonſtration of the Conſtruction by which the Area of the Ciſſoid is determined in the Third Example. Let the lines marked with points in the Scheme be expunged, draw the Chord DQ, and the A-



ſymptote QR, of the Ciſſoid. Then from the nature of the Circle, it is $DQq = AQ \times CQ$, and thence (by Prob. I.) 2DQ multiplied by the Fluxion of $DQ = AQ = AQ \times \dot{CQ}$ therefore $AQ : DQ :: 2\dot{DQ} : \dot{CQ}$. Alſo from the nature of the Ciſſoid it is $ED : AD :: AQ : DQ$; therefore $ED : AD :: 2\dot{DQ} : \dot{CQ}$; and $ED \times \dot{CQ} = AD \times 2\dot{DQ}$ or $4 \times \frac{1}{2} AD \times \dot{DQ}$: Now ſince DQ is perpendicular at the end of AD revolving about A; and $\frac{1}{2} AD \times 2\dot{DQ}$ is equal to the Fluxion generating the

the Area ADOQ; its quadruple alſo ED × CQ is equal to the Fluxion generating the Ciſſoidal Area QREDO. Wherefore that Area QREDO infinitely long, is generated quadruple of the other ADOQ. Q. E. D.

Scholium.

By the foreging Tables not only the Areas of Curves, but alſo Quantities of any other kind that are generated by an analogous way of Flowing may be derived from their Fluxions, and that by the aſſiſtance of this Theorem:

That a quantity of any kind is to an Unit of the ſame kind, as the Area of a Curve is to a ſuperficial Unity; if ſo be that the Fluxion generating that quantity be to an Unit of its kind, as the Fluxion generating the Area is to an Unit of its kind alſo; that is, as the Right Line moving perpendicularly upon the Abſciſs (or the Ordinate) by which the Area is deſcribed, to a linear Unit.

Wherefore if any Fluxion whatever is expounded by ſuch a moving Ordinate, the quantity generated by that Fluxion will be expounded by the Area deſcribed by ſuch Ordinate. Or if the Fluxion be expounded by the ſame algebraick terms as the Ordinate, the generating quantity will be expounded by the ſame as the deſcribed Area. Therefore the Equation, which exhibits a Fluxion of any kind, is to be ſought for in the firſt column of the Tables, and the value of t in the laſt column will ſhew the generated Quantity.

As

As if $\sqrt{1+\frac{9z}{4a}}$ exhibits a Fluxion of any kind, make it equal to y; and that it may be reduced to to the form of the equations in the Tables, substitute z^n for z, and it will be $z^{n-1}\sqrt{1+\frac{9}{4a}z^n}=y$, an equation of the first species of the Third Order of the first Table. And comparing the terms it will be $d=1$, $e=1$, $f=\frac{9}{4a}$; and thence $\frac{8a+18z}{27}\sqrt{1+\frac{9z}{4a}}=\frac{2d}{3f}R^3=t$. Therefore it is the quantity $\frac{8a+18z}{27}\sqrt{1+\frac{9z}{4a}}$ which is generated by the Fluxion $\sqrt{1+\frac{9z}{4a}}$.

And thus if $\sqrt{1+\frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ represents a Fluxion, by a due reduction (or by extracting $z^{\frac{2}{3}}$ out of the Radical, and writing z^n for $z^{-\frac{2}{3}}$, there will be had $\frac{1}{z^{n+1}}\sqrt{z^n+\frac{16}{9a^{\frac{2}{3}}}}=y$; an equation of the second species of the Fifth Order of the second Table. Then comparing the terms it is $d=1$, $e=\frac{16}{9a^{\frac{2}{3}}}$, and $f=1$. So that $z^{\frac{2}{3}}=\frac{1}{z^n}=xx$, $\sqrt{1+\frac{16xx}{9a^{\frac{2}{3}}}}=v$, and $\frac{1}{2}s=\frac{-2d}{n}s=t$; which being found, the quantity generated by the Fluxion $\sqrt{1+\frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ will be known, by making it to be to an unit of its own kind, as the Area $\frac{1}{2}s$ is to superficial unity; or which comes to to the ſame, by ſuppoſing the quantity t no longer to repreſent a ſuperficies, but a quantity of another kind, which is to an unit of its own kind, as that ſuperficies is to ſuperficial unity.

Thus, ſuppoſing $\sqrt{1+\frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ to repreſent a Linear Fluxion. I imagine t no longer to ſignify a Superficies, but a Line; that Line, for inſtance, which is to a Linear Unit, as the Area which (according to the Tables) is repreſented by t, is to a Superficial Unit, or that which is produced by applying that Area to a Linear Unit: on which account if that Linear Unit be made e, the Length generated by the foregoing Fluxion will be $\frac{3s}{2e}$. Upon this Foundation theſe Tables may be applied to the determining of LENGTHS of CURVES; the CONTENTS of their SOLIDS; and any other QUANTITIES whatever, as well as the Areas of Curves.

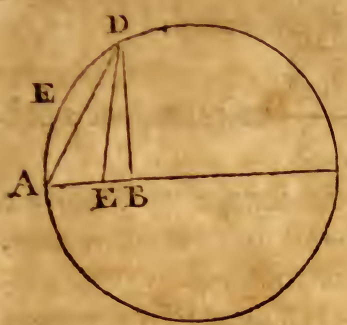
Of Queſtions that are related hereto.

1\. *To approximate to the Areas of Curves mechanically.*

The Method is this. That the Values of two or more Right lined Figures may be ſo compounded together, that they may very nearly conſtitute the value of the Curvilinear Area required.

Thus for the Circle AFD which is denoted by the Equation $x-xx=zz$. Having found the value of the Area AFDB, *viz.* $\frac{2}{3}z^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}-\frac{1}{72}x^{\frac{9}{2}}$, *&c.* the values of ſome rectangles are to be ſought, ſuch as the value $x\sqrt{x-xx}$ or $x^{\frac{3}{2}}-\frac{1}{2}x^{\frac{5}{2}}-\frac{1}{8}x^{\frac{7}{2}}-\frac{1}{16}x^{\frac{9}{2}}$, *&c.* of the rectangle BD × AB; and $x\sqrt{x}$ or

$x^{\frac{3}{2}}$ the value of AD × AB. Then these values are to be multiplied by any different letters, that stand for numbers indefinitely, and then to be added together; and the terms of the sum are to be compared with the corresponding terms of the value of the Area AFDB, that they may be made as nearly equal as possible. ——As if these Parallelograms were multiplied by e and f, the sum would be $\begin{matrix} ex^{\frac{3}{2}} \\ +f \end{matrix}$ $-\frac{1}{2}ex^{\frac{5}{2}} - \frac{1}{8}ex^{\frac{7}{2}}$, *&c.* the terms of which being compared with these terms $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}}$, *&c.* there arises $e+f=\frac{2}{3}$; $-\frac{1}{2}e=-\frac{1}{5}$, or $e=\frac{2}{5}$; and $f=\frac{2}{3}-e=\frac{4}{15}$. So that $\frac{2}{5}$BD × AB $+\frac{4}{15}$AD × AB $=$ Area AFDB very nearly: for $\frac{2}{5}$BD × AB $+\frac{4}{15}$AD × AB is equivalent to $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{20}x^{\frac{7}{2}} - \frac{1}{40}x^{\frac{9}{2}}$, *&c.* which being subtracted from the Area AFDB becomes the Error only $\frac{1}{70}x^{\frac{7}{2}} + \frac{1}{90}x^{\frac{9}{2}}$, *&c.*



Thus if AB were bisected in E, the value of the Rectangle AB × DE will be $x\sqrt{\overline{-\frac{3}{4}xx}}$, or $x^{\frac{3}{2}} - \frac{3}{8}x^{\frac{5}{2}} - \frac{9}{28}x^{\frac{7}{2}} - \frac{27}{1024}x^{\frac{9}{2}}$, *&c.* and this compared with the Rectangle AD × AB gives $\frac{8\text{DE}+2\text{AD}}{15}$ into AB $=$ Area AFDB; the error being only $\frac{1}{560}x^{\frac{3}{2}} + \frac{1}{5760}x^{\frac{5}{2}}$, *&c.* which is always less than $\frac{1}{1500}$th part of the whole Area; even though AFDB were a Quadrant of a Circle. But this Theorem may be thus propounded: As 3 to 2 so is the Rectangle AB × DE added to $\frac{1}{5}$th part of the difference between AD and DE, to the Area AFDB very nearly.

And

And thus by compounding two Rectangles AB × ED and AB × BD, or all the Rectangles, together, or by taking ſtill more Rectangles, other Rules may be invented, which will be ſo much more exact, as there are more rectangles made uſe of. And the ſame may be underſtood of the Area of the Hyperbola, or of any other Curves: nay, by only one Rectangle the Area may be very commodiouſly exhibited; as in the foregoing Circle by taking BE to AB as $\surd 10$ to 5, the Rectangle AB × ED will be to the Area AFDB as 3 to 2, the error being only $\frac{1}{175}x^{\frac{7}{2}} + \frac{11}{2250}x^{\frac{9}{2}}$, &c.

II. *The Area being given, to determine the Abſciſs and Ordinate.*

When the Area is expreſſed by a Finite Equation there can be no difficulty; but when it is expreſſed by an infinite ſeries, the affected root is to be extracted which denotes the Abſciſs. So for the Hyperbola defined by this Equation $\frac{ab}{a+x} = \dot{z}$; after you have found $z = bx - \frac{bx^2}{2a} + \frac{bx^3}{3a^2} - \frac{bx^4}{4a^3}$, &c, that from the given Area the Abſciſs x may be known, extract the affected root, and there will ariſe $x = \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{96a^4b^5}$, &c. And moreover, if the Ordinate $\dot{z}$ were required, divide ab by $a+x$, that is by $a + \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3}$, &c. and there will ariſe $\dot{z} = b - \frac{z}{b} - \frac{z^2}{2a^2b} - \frac{z^3}{6a^3b^2} - \frac{z^4}{24a^4b^3}$, &c.

Thus in the Ellypsis which is expressed by the Equation $ax - \frac{a}{c}xx = \dot{z}\dot{z}$, after the Area $z = \frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{a^{\frac{1}{2}}x^{\frac{5}{2}}}{5c} - \frac{a^{\frac{1}{2}}x^{\frac{7}{2}}}{28c^2} - \frac{a^{\frac{1}{2}}x^{\frac{9}{2}}}{72c^3}$, &c. is found, write v^3 for $\frac{3z}{2a^{\frac{1}{2}}}$, and t for $x^{\frac{1}{2}}$, and it will be $v^3 = t^3 - \frac{3t^5}{10c} - \frac{3t^7}{56c^2} - \frac{t^9}{48c^3}$, &c. and extracting the root, $t = v + \frac{v^3}{10c} + \frac{81v^5}{1400c^2} + \frac{1171v^7}{25200c^3}$, &c. whose square $v^2 + \frac{v^4}{5c} + \frac{22v^6}{175c^2} + \frac{823v^8}{7875c^3}$, &c. $= x$; this value being substituted instead of x in the equation $xx - \frac{a}{c}xx = \dot{z}\dot{z}$, and the root being extracted, there arises $\dot{z} = a^{\frac{1}{2}}v - \frac{2a^{\frac{1}{2}}v^3}{5c} - \frac{38a^{\frac{1}{2}}v^5}{175c^2} - \frac{407a^{\frac{1}{2}}v^7}{2250c^3}$, &c. So that from z, the given Area, and thence v or $\sqrt[3]{\frac{3z}{2a^{\frac{1}{2}}}}$, the Abscis x will be given, and the Ordinate $\dot{z}$. All which things may be accommodated to the Hyperbola, if only the sign of the quantity c be changed, wherever it is found of odd dimensions.

PROBLEM X.

To find as many Curves as we please, whose Lengths may be expressed by Finite Equations.

The following Positions prepare the way for the Solution of this Problem.

1. If the Right Line DC, standing perpendicularly upon any Curve AD, be conceived thus to move, all its points G, g, r, &c. will describe other

other Curves which are equi-diſtant and perpendicular to that Line: as GK, *gk*, *rs*, &c.

2. If that right line be continued indefinitely each way, its extremities will move contrary ways; and therefore there will be a point between, which will have no motion, but may therefore be called the Center of motion. This point will be the ſame as the center of curvature, which the Curve AD hath at the point D; as is mentioned before. Let that Point be C.

3. If we ſuppoſe the line AD not to be circular, but inequably curved; ſuppoſe more curved towards δ, and leſs towards Δ; that Center will continually change its place, approaching nearer to the parts more curved as in K, and going farther off at the parts leſs curved, as in *k*; and by that means will deſcribe ſome line as K C *k*.

4. The right line DC will continually touch the line deſcribed by the center of curvature. For if the point D of this line moves towards δ, its point G, which in the mean time paſſes to K, and is ſituate on the ſame ſide of the center C, will move the ſame way by the ſecond poſition. Again, if the ſame point D moves towards Δ, the point *g*, which in the mean time paſſes to *k*. and is ſituate on the contrary ſide of the center C, will move the contrary way; that

that is, the fame way that G moved in the former Cafe while it paffed to K; wherefore K and *k* lie on the fame fide of the right line DC; but as K and *k* are taken indefinitely for any points, it's plain that the whole Curve lies on the fame fide of the right line DC, and therefore is not cut but only touched by it.

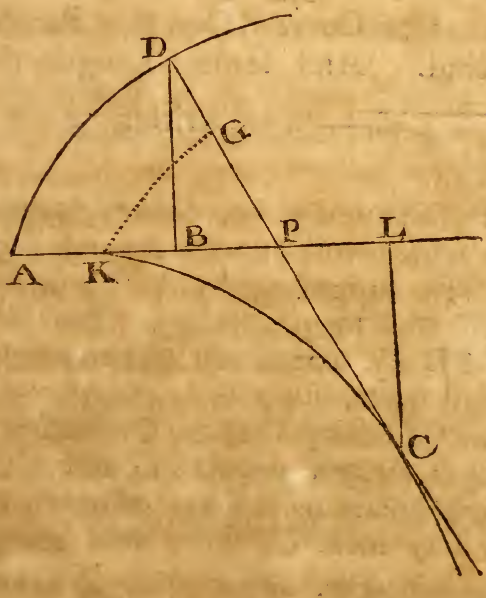
Here it is fuppofed that the line δDΔ is continually more curved towards δ, and lefs towards Δ; for if its greateft or leaft curvature is in D, then the right line DC will cut the curve KC, but yet in an angle that is lefs than any right-lined angle, which is the fame thing as if it were faid to touch it; Nay the point C in this cafe is the limit or Cufpid, at which the Two parts of the Curve, finifhing in the moft oblique concourfe, touch each other; and therefore may more juftly be faid to be touched, than to be cut, by the right line DC, which divides that angle of contact.

5. The right line CG is equal to the Curve CK. For conceive all the points *r*, 2*r*, 3*r*, 4*r*, &c. of that line to defcribe the arches of Curves *rs*, 2*r*2*s*, 3*r*3*s*, &c. in the mean time that they approach to the Curve CK by the motion of that right line; and fince thefe arches (by the firft pofition,) are perpendicular to the right lines that touch the Curve CK, (by the 4th pofition,) it follows that they will be alfo perpendicular to that Curve. Wherefore the parts of the line CK intercepted between thefe arches, which by reafon of their infinite fhortnefs may be confidered as right lines, are equal to fo many parts of the right line CG, and equals being added to equals the whole line CK will be equal to the whole line CG.

This would likewife appear by conceiving that every part of the right line CG, as it moves along, will apply itfelf fucceffively to every part of the Curve CK; and thereby will meafure ſure thoſe parts; juſt as the circumference of a Wheel, while it moves forwards by revolving upon a plain, will meaſure the diſtance that the point of contact continually deſcribes.

And hence it appears, that the Problem may be reſolved, by aſſuming any Curve at pleaſure AδDΔ and thence by determining the other Curve KCk, in which the center of curvature of the aſſumed Curve is always found. Therefore letting fall the Perpendiculars DB and CL to a right line AB given in poſition, and in AB taking any point A, and calling AB$=x$ and BD$=y$; to define the Curve AD let any relation be aſſumed between x and y, and then by PROB. V. the point C may be found; by which may be determined both the Curve KC, and its length GC.

Ex. Let $ax=yy$ be the equation to the Curve; which therefore will be the *Appollonian* Parabola. By



PROB. V. will be found AL$=\frac{1}{2}a+3x$, CL$=\frac{4y^3}{aa}$,

and

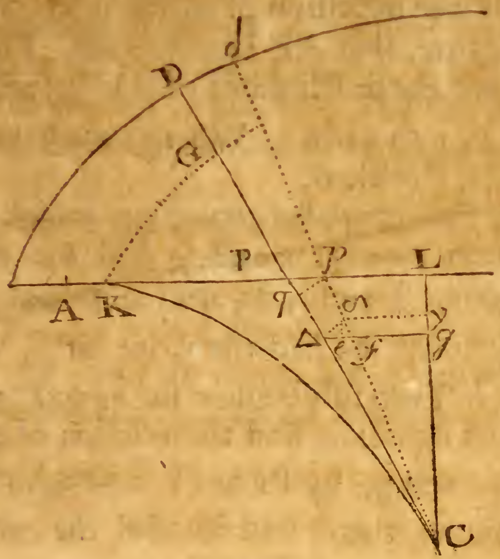
and DC$=\frac{a+4x}{a}\sqrt{\frac{1}{4}ax+ax}$; which being obtained, the Curve KC is determined by AL and LC, and its Length by DC. For as we are at liberty to assume the points K and C any where in the Curve KC, let us suppose K to be the center of curvature of the Parabola at its vertex; putting therefore AB and BD, or x and y, to be nothing, it will be DC$=\frac{1}{2}a$, and this is the Length AK or DG, which being subtracted from the former indefinite value of DC, leaves GC or KC$=\frac{a+4x}{a}\sqrt{\frac{1}{4}aa+ax}-\frac{1}{2}a$.

Now if you desire to know what Curve that is, and what is its Length, without any relation to the Parabola, call KL$=z$, and LC$=v$, then it will be $z=$AL$-\frac{1}{2}a=3x$; or $\frac{1}{3}z=x$, and $\frac{az}{3}=ax=yy$, therefore $4\sqrt{\frac{z^3}{27a}}=\frac{4y^3}{aa}=CL=v$; or $\frac{16z^3}{27a}=v^2$, which shews the Curve KC to be a Parabola of the second Kind. And for its Length there arises $\frac{3a+4z}{3a}\sqrt{\frac{1}{4}aa+\frac{1}{3}az}-\frac{1}{2}a$ by writing $\frac{1}{3}z$ for x in the value of CG.

The Problem may be resolved by taking any equation which will express the relation between AP and PD, supposing P to be the intersection of the Abscis and Perpendicular. For calling AP $=x$ and PD$=y$, conceive CPD to move an infinitely small space, suppose to the place Cpd; and in CD and Cd taking CΔ and Cδ both of the same given length, suppose equal to 1; and to CL let fall the Perpendiculars Δg and δy, of which Δg (which call $=z$) may meet Cd in f; then compleat the Parallelogram $g y \delta e$, and making $\dot{x}$, $\dot{y}$, and $\dot{z}$, the Fluxions of the quantities x, y, and z, as before; it will be $\Delta e \,.\, \Delta f :: \overline{\Delta e}^2 : \overline{\Delta\delta}^2 :: \overline{CG}^2 : \overline{C\Delta}^2 :: \frac{\overline{Cg}^2}{C\Delta} : C\Delta$.

And

And $\Delta f : Pp :: C\Delta : CP$. Then *ex æquo* $\Delta e : Pp :: \frac{\overline{Cg}^2}{C\Delta} : CP$. But Pp is the moment of the Abſciſs



AP, by the acceſſion of which it becomes Ap; and Δe is the contemporaneous moment of the Perpendicular Δg, by the decreaſe of which it becomes δy; therefore Δe and Pp, are as the Fluxions of the lines Δg (z) and AP (x); that is, as $\dot{z}$ and $\dot{x}$. Wherefore $\dot{z} : \dot{x} :: \frac{\overline{Cg}^2}{C\Delta} : CP$. And ſince it is $\overline{CG}^2 = \overline{C\Delta}^2 - \overline{\Delta g}^2 = 1 - zz$; and $C\Delta = 1$; it will be $CP = \frac{\dot{x} - \dot{x}z^2}{\dot{z}}$. Moreover ſince we may aſſume any one of the Three $\dot{x}$, $\dot{y}$, $\dot{z}$, for an uniform Fluxion to which the reſt may be referred; if $\dot{x}$ be that Fluxion, and its value be unity; then $CP = \frac{1 - zz}{\dot{z}}$.

Beſides it is $C\Delta$ $(1) : \Delta g$ $(z) :: CP : PL$; alſo $C\Delta$ $(1) : Cg$ $(\sqrt{1 - zz}) :: CP : CL$. Therefore it is $PL = \frac{z - z^3}{\dot{z}}$; and $CL = \frac{1 - zz}{\dot{z}} \sqrt{1 - zz}$. Laſtly, drawing *pq* parallel to the infinitely ſmall arch D*d*, or perpendicular to DC, P*q* will be the *momentum* of DP by the acceſſion of which it becomes *dp*, at the ſame time that AP becomes A*p*. Therefore P*p* and P*q* are as the Fluxions of AP (x) and PD (y); that is, as 1 to $\dot{y} = z$. Whence we have this ſolution of the Problem.

From the propoſed equation which expreſſes the relation between x and y find the relation of the Fluxions $\dot{x}$ and $\dot{y}$ by Prob. I. and putting $\dot{x} = 1$, there will be had the value of $\dot{y}$, to which z is equal. Then ſubſtitute z for $\dot{y}$, and by the help of the laſt equation find the relation of the Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$, by Prob. I. and again ſubſtituting 1 for $\dot{x}$, there will be had the value of $\dot{z}$. Theſe being found make $\frac{1 - \dot{y}\dot{y}}{\dot{z}} = CP$, $z \times CP = PL$, and $CP\sqrt{1 - \dot{y}\dot{y}} = CL$, and C will be a point in the Curve; any part of which KC is equal to a right line CG, which is the difference of the Tangents drawn perpendicularly to D*d* from C and K.

Example. Let $xx = yy$, be the equation which expreſſes the relation between AP and PD; and by Prob. I. it will be firſt $a\dot{x} = 2y\dot{y}$, or $a = 2yz$: Then $0 = 2\dot{y}z + 2y\dot{z}$, or $\frac{-zz}{y} = \dot{z}$. Thence it is $CP = \frac{1 - \dot{y}\dot{y}}{\dot{z}} = y - \frac{4y^3}{aa}$, $PL = z \times CP = \frac{1}{2}a - \frac{2yy}{a}$, and $CL = \frac{aa - 4yy}{2aa}\sqrt{4yy - aa}$. And from CP and PL taking away y and x, there remains $CD = -\frac{4y^3}{aa}$ and $AL =$

$\frac{1}{2}a -$

$\frac{1}{2}a-\frac{3yy}{a}$. Now I take away y and x, because when CP and PL have affirmative values, they fall on the fide of the point P towards D and A, and they ought to be diminifhed by taking away the affirmative quantities PD and AP; but when they have negative values, they will fall on the contrary fide of the point P, and they muft be increafed, which is alfo done by taking away the affirmative quantities PD and AP.

Now to know the length of the Curve in which the point C is found between any two of its points K and C; we muft feek the lengths of the Tangent at the point K and fubtract it from CD. As if K were the point at which the Tangent is terminated, when CΔ and Δg or 1 and z are made equal, which therefore is fituated in the Abfcifs itfelf AP; write 1 for z in the equation $a=2yz$, whence $a=2y$. Therefore for y write $\frac{1}{2}a$ in the value of CD that is in $\frac{-4y^3}{aa}$, and it comes out $-\frac{1}{2}a$; and this is the length of the Tangent at the point K, or of DG; the difference between which and the foregoing indefinite value of CD is $\frac{4y^3}{aa}-\frac{1}{2}a$, that is GC, to which the part of the Curve KC is equal.

Now that it may appear what Curve that is, from AL (having firft changed its fign, that it may become affirmative) take AK, which will be $\frac{1}{4}a$, and there will remain $KL=\frac{3yy}{a}-\frac{3}{4}a$, which call t; and in the value of the line CL, which call v write $\frac{4at}{3}$, for $4yy-aa$; and there will arife $\frac{2t}{3a}\sqrt{\frac{4}{3}at}=v$ or $\frac{16t^3}{27a}=vv$, which is an equation to a Parabola of the fecond Kind; as was found before.

When the relation between t and v cannot conveniently be reduced to an equation, it may be sufficient only to find the length PC and PL. As if for the relation between AP and PD the equation $3a^2x+3a^2y-y^3=0$ were assumed. From hence, by PROB. I. first there arises $a^2+a^2z-y^2z=0$, then $aa\dot{z}-2y\dot{y}z-y^2\dot{z}=0$, and therefore it is $z=\frac{aa}{yy-aa}$, $\dot{y}$ and $\dot{z}=\frac{2y\dot{y}z}{aa-yy}$. Whence are given $PC=\frac{1-yy}{\ddot{z}}$, and $PL=z\times PC$; by which the point C is determined, which is in the Curve. And the length of the Curve, between Two such points, will be known, by the difference of the Two corresponding Tangents DC or PC$-y$.

For Example. If we make $a=1$, and in order to determine some point of the Curve C, we take $y=2$; then AP or x becomes $\frac{y^3-3a^2y}{3aa}=\frac{2}{3}$, $z=\frac{1}{3}$, $\dot{z}=-\frac{4}{9}$, PC$=-2$, and PL$=-\frac{2}{3}$. Then to determine another point if we take $y=3$; it will be AP$=6$, $z=\frac{1}{8}$, $\dot{z}=-\frac{3}{256}$, PC$=-84$, and PL$=-10\frac{1}{2}$; which being had, if y be taken from PC, there will remain -4 in the first case, and -87 in the second, for the length DC; the difference of which 83 is the Length of the Curve between the two points found C and c.

These are to be thus understood, when the Curve is continued between the two points C and c, or between K and C, without any term or limit, which we called its Cuspid. For when one or more such terms come between these points (which terms are found by the determination of the greatest or least PC or DC), the Lengths of each of the parts of the Curve between them and the points C or K must be separately found, and then added together.

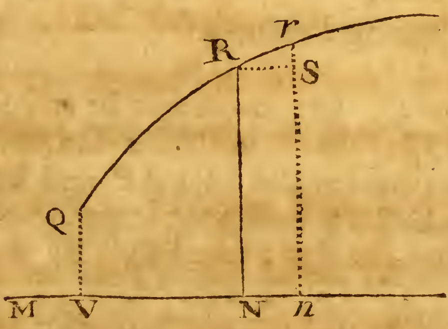
PROBLEM

PROBLEM XI.

To find as many Curves as you please, whose Lengths may be compared with the Length of any Curve proposed, or with the Area applied to a given Line by the help of Finite Equations.

It is performed by involving the length or the Area of the proposed Curve in the equation, which is assumed in the foregoing Problem, to determine the relation between AP and PD; [*Fig. p.* 143.] but that z and $\dot{z}$ may be thence derived by PROB. I. the Fluxion of the length or of the Area must be first discovered.

The Fluxion of the Length is discovered by putting it equal to the square root of the sum of the squares of the Fluxion of the Abscifs and of the Ordinate. For let RN be the per-



pendicular Ordinate moving upon the Abscifs MN. And let QR be the proposed Curve, at which RN is terminated. Then calling MN $=s$, NR$=t$, and QR$=v$, and their Fluxions $\dot{s}$, $\dot{t}$, and $\dot{v}$, respectively; conceive the line NR to move into the place nr infinitely near the former, and letting fall Rs perpendicularly to nr; then

then Rs, sr, and Rr, will be the contemporaneous moments of the lines MN, NR, and QR, by the accession of which they become Mn, nr, and Qr; but as these are to each other as the Fluxions of the same lines, and because of the Rectangle Rsr, it will be $\sqrt{\overline{Rs}^2 + \overline{sr}^2} = Rr$, or $\sqrt{\dot{s}^2 + \dot{t}^2} = \dot{v}$.

But to determine the Fluxions $\dot{s}$ and $\dot{t}$, there are two equations required: One of which is to define the relation between MN and NR or s and t, from whence the relation between the Fluxions $\dot{s}$ and $\dot{t}$ may be derived: And another which may define the relation between MN or NR in the given figure, and of AP or x in that required, from whence the relation of the Fluxion $\dot{s}$ or $\dot{t}$ to the Fluxion $\dot{x}$ or 1 may be discovered.

Then $\dot{z}$ being found, the Fluxions $\dot{y}$ and $\dot{z}$ may be sought by a Third assumed Equation, by which the length PD or y may be defined. Then we are to take $PC = \frac{1 - \dot{y}\dot{y}}{\dot{z}}$, $PL = \dot{y} \times PC$, and $DC = PC - y$, as in the foregoing Problem.

EXAMPLE I. Let $as - ss = tt$ be an equation to the Curve QR, which will be a Circle; $xx = as$ the relation between the lines AP and MN; and $\frac{2}{3}v = y$ the relation between the length of the Curve given QR and the right line PD. By the first it will be $a\dot{s} - 2s\dot{s} = 2t\dot{t}$ or $\frac{a - 2s}{2t}\dot{s} = \dot{t}$, and thence $\frac{as}{2t} = \sqrt{\dot{s}^2 + \dot{t}^2} = \dot{v}$; by the second it is $2x = a\dot{s}$, and therefore $\frac{x}{t} = \dot{v}$; and by the third $\frac{2}{3}\dot{v} = \dot{y}$, that is, $\frac{2x}{3t} = z$, and hence $\frac{2}{3t} - \frac{2x\dot{t}}{3tt} = \dot{z}$, which being found, take $PC = \frac{1 - \dot{y}\dot{y}}{\dot{z}}$, $PL = \dot{y} \times PC$, and DC $=$ PC

$=PC-y$, or $PC-\frac{2}{3}QR$. Where it appears, that the Length of the given Curve QR cannot be found, but at the same time the Length of the right Line DC must be known, and from thence the Length of the Curve in which the point C is found. And *vice versâ*.

Ex. 2. The equation $as-ss=tt$ remaining, make $x=s$, and $vv-4ax=4ay$; and by the first three will be found $\frac{a\dot{s}}{2t}=\dot{v}$ as above; but by the second $1=\dot{s}$, and therefore $\frac{a}{2t}=\dot{v}$, and by the third $2v\dot{v}-4a=4a\dot{y}$ (or eliminating $\dot{v}$) $\frac{v}{4t}-1=\dot{z}$. And from hence $\frac{\dot{v}}{4t}-\frac{v\dot{t}}{4tt}=\dot{z}$.

Ex. 3. Let there be supposed three equations, $aa=st$, $a+3s=x$, and $x+v=y$. Then by the first (which denotes an Hyperbola) it is $0=\dot{s}t+\dot{t}s$, or $-\frac{\dot{s}t}{s}=\dot{t}$, and therefore $\frac{\dot{s}}{s}\sqrt{ss+tt}=\sqrt{\dot{s}\dot{s}+\dot{t}\dot{t}}=\dot{v}$; by the second it is $3\dot{s}=1$, and therefore $\frac{1}{3s}\sqrt{ss+tt}=\dot{v}$; and by the third it is $1+\dot{v}=\dot{y}$, or $1+\frac{1}{3s}\sqrt{ss+tt}=\dot{z}$. Thence it is $\dot{w}=\dot{z}$, that is, putting $\dot{w}$ for the Fluxion of the radical $\frac{1}{3s}\sqrt{ss+tt}$, which if it be made equal to w, or $\frac{1}{9}+\frac{tt}{9ss}=ww$ there will arise from hence $\frac{2t\dot{t}}{9ss}-\frac{2tt\dot{s}}{9s^3}=2w\dot{w}$, and first substituting $\frac{-\dot{s}t}{s}$ for $\dot{t}$, then $\frac{1}{3}$ for $\dot{s}$, and dividing by $2w$, there will arise $\frac{-2tt}{27ws^3}=\dot{w}=\dot{z}$. Now $\dot{y}$ and $\dot{z}$ being found, the rest is performed as in the first example.

Now

Now if from any point of a Curve Q a perpendicular QV be let fall on MN, and a Curve is to be found, whose length may be known from the length which arises by applying the Area QRNV to any given line; let that given line be called E, the length $\frac{QRNV}{E}$ which is produced by such application be called v, and its Fluxion $\dot{v}$; and since the Fluxion of the Area QRNV is to the Fluxion of the Area or Rectangular Parallelogram made upon VN with the height E, as the Ordinate or moving line $NR = t$, by which this is described, to the moving line E, by which the other is described in the same time; and the Fluxions $\dot{v}$ and $\dot{s}$ of the lines v and MN, (or s,) or of the lengths which arise by applying these areas to the given line E, are in the same ratio; it will be $\dot{v} = \frac{\dot{s}t}{E}$. Therefore by this rule the value of $\dot{v}$ may be sought, and the rest to be performed as in the examples foregoing.

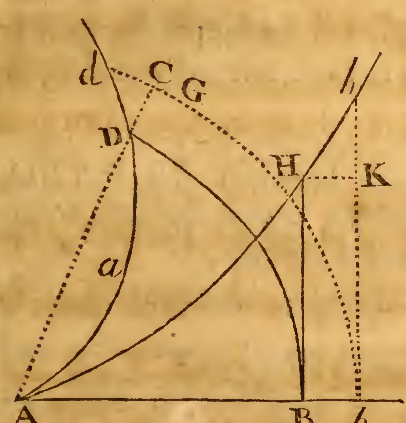
Ex. 4. Let QR be an Hyperbola which is defined by this equation $aa + \frac{ass}{c} = tt$; thence arises by PROB. I. $\frac{a\dot{s}s}{c} = t\dot{t}$, or $\frac{a\dot{s}s}{ct} = \dot{t}$. Then if for the other two equations are assumed $x = s$, and $y = v$, the first will give $1 = \dot{s}$, whence $\dot{v} = \frac{\dot{s}t}{E} = \frac{t}{E}$; and the latter will give $\dot{y} = \dot{v}$, or $z = \frac{t}{E}$; from hence $\dot{z} = \frac{\dot{t}}{E}$, and substituting $\frac{a\dot{s}s}{ct}$ or $\frac{as}{ct}$ for $\dot{t}$, it becomes $\dot{z} = \frac{as}{ctE}$. Now $\dot{y}$ and $\dot{z}$ being found, make $\frac{1-\dot{y}\dot{y}}{\dot{z}} = CP$, and $\dot{y} \times CP = PL$, as before; and thence

the

the point C will be determined, and the Curve in which all ſuch points are ſituated: The length of which Curve will be known from the length DC, which is equivalent to CP—v, as is ſufficiently ſhewn before.

There is alſo another method, by which the Problem may be reſolved, and that is by finding Curves whoſe Fluxions are either equal to the Fluxion of the propoſed Curve, or are compounded of the Fluxions of that and of other lines; and this may ſometimes be of uſe in converting Mechanical Curves into equable Geometrical Curves, of which thing there is a remarkable example in *Spiral* Lines.

Let AB be a right line given in poſition, BD an arch moving upon AB as an Abſciſs, and yet retaining A as its center; AD*d* a Spiral at which that arch is continually terminated, *bd* an arch indefinitely near it, or the place into which the arch BD by its motion next arrives; DC a perpendicular to the arch *bd*, *d*G the difference of the arches; AH another



Curve equal to the Spiral AD; BH a right line moving perpendicularly upon AB, and terminated at the Curve AH; *bh* the next place into which that right line moves; and HK perpendicular to *bh*; and in the infinitely little Triangles DC*d* and HK*h*, ſince DC and HK are equal to ſome third line B*b*, and therefore equal to each other; and D*d* and H*h* (by Hypotheſis,) are correſpondent parts of equal Curves, and therefore equal

 to

to each other, as alſo the angles at C and K are right angles; the third ſides dC and bK will be equal alſo. Moreover ſince it is $AB : BD :: Ab : bC :: Ab - AB\ (Bb) : bC - BD\ (CG)$; therefore $\frac{BD \times Bb}{AB} = CG$; if this be taken from dG, there will remain $dG - \frac{BD \times Bb}{AB} = dC = bK$. Call therefore $AB = z$, $BD = v$, and $BH = y$, and their Fluxions $\dot{z}$, $\dot{v}$, and $\dot{y}$, reſpectively, ſince Bb, dG, bK, are the contemporaneous moments of the ſame, by the acceſſion of which they become, Ab, bd, and bb; and therefore are to each other as the Fluxions. Therefore for the moments in the laſt equation let the Fluxions be ſubſtituted as alſo the letters for the lines, and there will ariſe $\dot{v} - \frac{v\dot{z}}{z} = \dot{y}$. Now, of theſe Fluxions, if $\dot{z}$ be ſuppoſed equable, or the unit to which the reſt are referred, the equation will be $\dot{v} - \frac{v}{z} = \dot{y}$.

Wherefore the relation between AB and BD, (or between z and v,) being given by any equation, by which the Spiral is defined, by PROB. I. the Fluxion $\dot{v}$ will be given; and thence alſo the Fluxion $\dot{y}$ by putting it equal to $\dot{v} - \frac{v}{z}$; and by PROB. II. this will give the line y, or BH, of which it is the Fluxion.

EX. 1. If the equation $\frac{zz}{a} = v$ were given (which is to the Spiral of *Archimedes*) thence (by PROB. I.) $\frac{2z}{a} = \dot{v}$; from this take $\frac{v}{z}$, or $\frac{z}{a}$, and there will remain $\frac{z}{a} = \dot{y}$, and thence by (PROB. II.) $\frac{zz}{2a} = y$; which ſhews the Curve AH, to which the Spiral AD is equal, to be the Parabola of *Apollonius*, who ſe whose *latus rectum* is $2a$, or whose Ordinate BH is always equal to $\frac{1}{2}$ Arch BD.

Ex. 2. If the Spiral be proposed which is defined by the equation $z^3 = av^2$, or $v = \frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}}$; there arises (by PROB. I.) $\frac{3z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{v}$, from which if you take $\frac{v}{z}$ or $\frac{z^{\frac{1}{2}}}{a^{\frac{1}{2}}}$, there will remain $\frac{\dot{z}^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{y}$, and thence (by PROB. II.) will be produced $\frac{z^{\frac{3}{2}}}{3a^{\frac{1}{2}}} = y$; that is $\frac{1}{3}$BD=BH, AH being a Parabola of the Second Kind.

Ex. 3. If the equation to the Spiral be $z\sqrt{\frac{a+z}{c}} = v$, then by PROB. I. $\frac{2a+3z}{2\sqrt{ac+cz}} = \dot{v}$, from which if you take $\frac{v}{z}$ or $\sqrt{\frac{a+z}{c}}$, there will remain $\frac{z}{2\sqrt{ac+cz}} = \dot{y}$. Now since the quantity generated by this Fluxion $\dot{y}$ cannot be found by PROB. II. unless it be resolved into an infinite Series; according to the tenor of the Scholium to PROB. IX. I reduce it to the form of the equations in the first column of the Tables, by substituting z^n for z, then it becomes $\frac{z^{2n-1}}{2\sqrt{ac+cz^n}} = \dot{y}$, which equation belongs to the second species of the fourth Order of the first Table; and by comparing the terms it is $d=\frac{1}{2}$, $e=ac$,

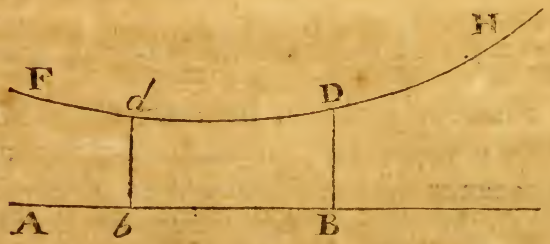
$e = ac$, and $f = c$, so that $\frac{z-2a}{3c}\sqrt{ac+cz} = t = y$, which equation belongs to a Geometrical Curve AH, that is equal in length to the Spiral AD.

PROBLEM XII.

To determine the Lengths of Curves.

In the foregoing Problem we have ſhewn that the Fluxion of a Curve Line is equal to the ſquare root of the ſum of the ſquares of the Fluxions of the Abſciſs and of the perpendicular Ordinate. Wherefore if we take the Fluxion of the Abſciſs for an uniform and determinate meaſure, or for an Unit, to which the other Fluxions may be referred; and alſo if from the equation which defines the Curve we find the Fluxion of the Ordinate, we ſhall have the Fluxion of the Curve Line from whence its Length may be deduced by the ſecond Problem.

EXAMPLE I. Let the Curve FDH be propoſed, which is defined by the equation $\frac{z^3}{aa} + \frac{aa}{12z} = y$;



making the Abſciſs $AB = z$, and the moving ordinate $DB = y$; then from the equation will be had, by PROB. I. $\frac{3zz}{aa} - \frac{aa}{12zz} = \dot{y}$, the Fluxion of z being 1, and $\dot{y}$ being the Fluxion of y; then adding the ſquares

squares of the Fluxions, the sum will be $\frac{9z^4}{a^4}+\frac{1}{2}+\frac{a^4}{144z^4}=\dot{t}\dot{t}$, and extracting the root $\frac{3zz}{aa}+\frac{aa}{12zz}=\dot{t}$, and thence by PROB. II. $\frac{z^3}{aa}-\frac{aa}{12z}=t$. Here $\dot{t}$ stands for the Fluxion of the Curve, and t for its length.

Therefore if the length dD of any portion of this Curve were required, from the points d and D, let fall the perpendiculars db and DB to AB, and in the value of t, substitute the quantities Ab and AB severally for z, and the difference of the results will be dD the length required. As if A$b=\frac{1}{2}a$, and AB$=a$, writing $\frac{1}{2}a$ for z, it becomes $t=\frac{-a}{24}$; but writing a for z, it becomes $t=\frac{11a}{12}$; from which, if the first value be taken, there will remain $\frac{23a}{24}$ for the length dD: or if only Ab be determined to be $\frac{1}{2}a$, and AB be looked upon as indefinite, there will remain $\frac{z^3}{aa}-\frac{aa}{12z}+\frac{a}{24}$ for the value of dD.

If you would know the portion of the Curve line which is represented by t; suppose the value of t to be $=0$, and there arises $z^4=\frac{a^4}{12}$, or $z=\frac{a}{\sqrt[4]{12}}$, therefore if you take A$b=\frac{a}{\sqrt[4]{12}}$, and erect the perpendicular bd, the length of the Arch dD will be t, or $\frac{z^3}{aa}-\frac{aa}{12z}$: And the same is to be understood of all Curves in general.

After the same manner by which we have determined the length of this Curve; if the equation $\frac{z^4}{a^3}+\frac{a^3}{32z^2}=y$ be proposed, for defining the nature

nature of another Curve; there will be deduced $\frac{z^4}{a^3} - \frac{a^3}{32z^2} = \dot{t}$. ——Or if this Equation be proposed $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} - \frac{1}{3}a^{\frac{1}{2}}z^{\frac{1}{2}} = y$; there will arise $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} + \frac{1}{3}a^{\frac{1}{2}}z^{\frac{1}{2}} = \dot{t}$. —Or in general if it be $cz^{\theta} + \frac{z^{2-\theta}}{4\theta\theta c - 8\theta c} = y$; (where θ is used for representing any number either integer or fraction;) We shall have $cz^{\theta} - \frac{z^{2-\theta}}{4\theta\theta c - 8\theta c} = \dot{t}$.

Ex. 2. Let the Curve be proposed, which is defined by this equation $\frac{2aa+2zz}{3aa}\sqrt{aa+zz} = y$; then (by PROB. I.) will be had $\dot{y} = \frac{4a^4z + 8a^2z^3 + 4z^5}{3a^4y}$; or exterminating y, $\dot{y} = \frac{2z}{aa}\sqrt{aa+zz}$; to the square of which add 1, and the sum will be $1 + \frac{4zz}{aa} + \frac{4z^4}{a^4}$; and its root $1 + \frac{2zz}{aa} = \dot{t}$: Hence by PROB. II. will be obtained $z + \frac{2z^3}{3a^2} = t$.

Ex. 3. Let a Parabola of the second Kind be proposed, whose equation is $z^3 = ay^2$, or $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} = y$; thence by PROB. I. is derived $\frac{3z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{y}$; therefore $\sqrt{1 + \frac{9z}{4a}} = \sqrt{1 + \dot{y}\dot{y}} = \dot{t}$. Now since the length of the Curve generated by the Fluxion $\dot{t}$ cannot be found by PROB. II. without a reduction to an infinite

finite ſeries of ſimple terms; I conſult the Tables at PROB. IX. and according to the Scholium belonging to it I have $t = \frac{8a+18z}{27}\sqrt{1+\frac{9z}{4a}}$.

And thus you may find the Length of theſe Parabolas $z^5=ay^4$, $z^7=ay^6$, $z^9=ay^8$, &c.

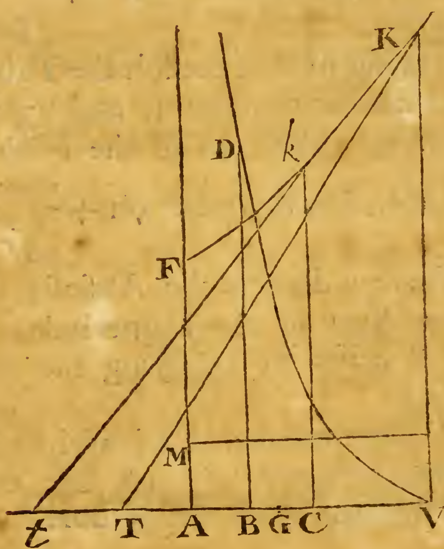
Ex. 4. Let the Parabola be propoſed whoſe equation is $z^4=ay^3$, or $\frac{z^{\frac{4}{3}}}{a^{\frac{1}{3}}}=y$; thence by PROB. I. will ariſe $\frac{4z^{\frac{1}{3}}}{3a^{\frac{1}{3}}}=\dot{y}$; therefore $\sqrt{1+\frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}=\sqrt{\dot{y}\dot{y}+1}=\dot{t}$. This being found, I conſult the Tables according to the aforeſaid Scholium, and by comparing with the ſecond Theorem of the Fifth Order of the latter Table, I have $z^{\frac{1}{3}}=x$, $\sqrt{1+\frac{16xx}{9a^{\frac{2}{3}}}}=v$, and $\frac{3}{2}s=t$. Where x denotes the Abſciſs, y the Ordinate, s the Area of the Hyperbola, and t the length which ariſes by applying the Area $\frac{3}{2}s$ to Linear Unity.

After the ſame manner the lengths of the Parabolas $z^6=ay^5$, $z^8=ay^7$, $z^{10}=ay^9$, &c. may alſo be reduced to the area of the Hyperbola.

Ex. 5. Let the Ciſſoid of the Antients be propoſed, whoſe equation is $\frac{aa-2az+zz}{\sqrt{az-zz}}=y$; thence (by PROB. I.) $\frac{-a-2z}{2zz}\sqrt{az-zz}=\dot{y}$; and therefore $\frac{a}{2z}\sqrt{\frac{a+3z}{z}}=\sqrt{\dot{y}\dot{y}+1}=\dot{t}$; which, by writing z^n for $\frac{1}{z}$ or z^{-1}, becomes $\frac{a}{2z}\sqrt{az^n+3}=\dot{t}$; an equation of the firſt ſpecies of the third Order of the Latter Table

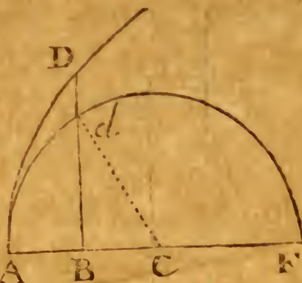
Table. Then comparing the terms it is $\frac{a}{z} = d$, $3 = e$, and $a = f$; so that $z = \frac{1}{z^n} = x^2$, $\sqrt{a + 3xx} = v$, and $6s - \frac{2v^3}{x} = \frac{4de}{vf}$ into $\frac{v^3}{2ex} - s = t$: and taking a for unity, by the multiplication or division of which, these quantities may be reduced to a just number of dimensions, it becomes $az = xx$, $\sqrt{aa + 3xx} = v$, and $\frac{6s}{a} - \frac{2v^3}{ax} = t$. Which are thus constructed.

The Cissoid being VD, AV the diameter of the circle to which it is adapted, AF its asymptote,



and DB perpendicular to AV cutting the Curve in D; with the semi-axis AF=AV, and the semi-parameter AG=$\frac{1}{3}$AV, let the Hyperbola F*k*K be described; and between AB and AV take AC a mean porportional, at C and V let the perpendiculars C*k* and VK be erected cutting the Hyperbola in *k* and K, and let the right lines *kt* and KT be drawn, touching the same in those points and meeting AV in *t* and T; and at AV let the rect-

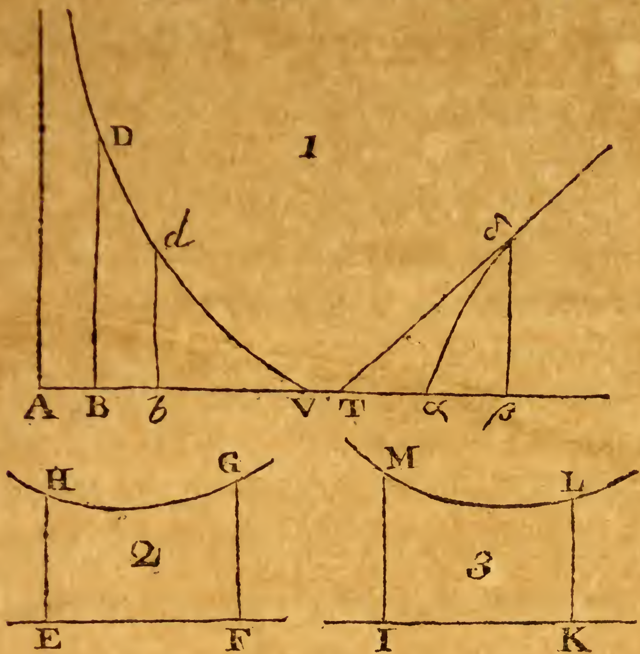
rectangle AVNM be described equal to the space TK*kt*. Then the length of the Cissoid VD will be the sextuple of the altitude VN.



Ex. 6. Supposing A*d* to be an Ellypsis, which the equation $\sqrt{az-2zz}=y$ represents; let the Mechanichal Curve AD be proposed of such a nature, that if D*d* or *y* be produced, till it meets this Curve at D, let BD be equal to the Elliptical arch A*d*. Now that the length of this may be determined, the equation $\sqrt{az-2zz}=y$, will give $\frac{a-4z}{2\sqrt{az-2zz}}=\dot{y}$, to the square of which if 1 be added, there arises $\frac{aa-4az+8zz}{4az-8zz}$, the square of the Fluxion of the arch A*d*; to which if 1 be added again, there will arise $\frac{aa}{4az-8zz}$, whose square root $\frac{a}{2\sqrt{az-2zz}}$ is the Fluxion of the Curve Line AD. Where if *z* be extracted out of the radical, and for z^{-1} be written z^{η}, there will be had $\frac{a}{2z\sqrt{az^{\eta}-2}}$, a Fluxion of the first species of the fourth Order of the latter Table: therefore the terms being collated there will arise $d=\frac{1}{2}a$, $e=-2$, and $f=a$; so that $z=\frac{1}{z^{\eta}}=x$, $\sqrt{ax-2xx}=v$, and $\frac{8s}{a}-\frac{4xv}{a}+v=\frac{8de}{rff}$ into $s-\frac{1}{2}xv-\frac{fv}{4e}=t$.

The Construction of which is thus. That the right line *d*C being drawn to the center of the Ellipsis, a Parallelogram be made upon AC= Sector AC*d*; then twice its height will be the length of the Curve AD.

Ex. 7. Making $A\beta = \varphi$, [*Fig.* 1.] and $\alpha\delta$ being an Hyperbola, whose equation is $\sqrt{-a+b\varphi\varphi}$



$=\beta\delta$, and its Tangent δT being drawn; let the Curve VdD be proposed, whose Abscis is $\frac{1}{\varphi\varphi}$; and its perpendicular ordinate is the length BD, which arises by applying the area $\alpha\delta T\alpha$ to Linear Unity. Now that VD the length of this may be determined, seek the Fluxion of the Area $\alpha\delta T\alpha$ when AB flows uniformly, and you will find this to be $\frac{a}{4bz}\sqrt{b-az}$, putting $AB=z$, and its Fluxion unity. For it is $AT=\frac{a}{b\varphi}=\frac{a}{b}\sqrt{z}$, and its Fluxion is $\frac{a}{2b\sqrt{z}}$, whose half drawn into the altitude $\beta\delta$ or $\sqrt{-a+\frac{b}{z}}$ is the Fluxion of the Area $\alpha\delta T$ described by the Tangent δT; therefore that Fluxion is $\frac{a}{4bz}\sqrt{b-az}$; and this applied to unity becomes

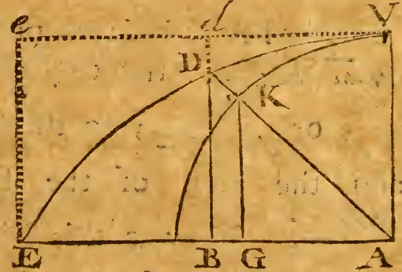
comes the Fluxion of the Ordinate BD; to the ſquare of this $\frac{aab-a^3z}{16b^2z^2}$ add 1 the ſquare of the Fluxion BD, and there ariſes $\frac{aab-a^3z+16b^2z^2}{16b^2z^2}$, whoſe root $\frac{1}{4bz}\sqrt{a^2b-a^3z+16b^2z^2}$ is the Fluxion of the Curve VD. But this is a Fluxion of the firſt ſpecies of the ſeventh Order of the latter Table; and the terms being collated, there will be $\frac{1}{4b}=d$, $aab=e$, $-a^3=f$, $16b^2=g$; and therefore $z=x$, and $\sqrt{a^2b-a^3x+16b^2x^2}=v$, (an equation to one Conick Section; ſuppoſe HG [*Fig.* 2.] whoſe area EFGH is s where EF$=x$, and FG$=v$,) alſo $\frac{1}{z}=\xi$, and $\sqrt{16bb-a^3\xi+ab\xi^2}=\Upsilon$, (an equation to another Conick Section; ſuppoſe ML [*Fig.* 3.] whoſe area IKLM is σ, where IK$=\xi$, and KL$=\Upsilon$.) Laſtly $\frac{2aabb\xi\Upsilon-a^3b\Upsilon-a^4v-4aabb\sigma-32abbs}{64b^4-a^4}=t$.

Wherefore that the length of any portion D*d* of the Curve VD may be known, let *db* be perpendicular to AB, and make A$b=z$; thence by what is now found ſeek the value of t; then make AB$=z$, and thence alſo ſeek for t; and the difference of theſe two Values of t will be the length D*d* required.

Ex. 8. Let the Hyperbola be propoſed, whoſe equation is $\sqrt{aa+bzz}=y$; thence (by PROB. I.) will be had $\dot{y}=\frac{bz}{y}$, or $\frac{bz}{\sqrt{aa+bzz}}$; to the ſquare of this add 1, and the root of the ſum will be $\sqrt{\frac{aa+bzz+bbzz}{aa+bzz}}=\dot{t}$. Now as this Fluxion is not to be found in the Tables, I reduce it to an infinite ſeries: And firſt by diviſion it becomes $\dot{t}=\sqrt{1+\frac{b^2}{a^2}z^2-\frac{b^3}{a^4}z^4+\frac{b^4}{a^6}z^6-\frac{b^5}{a^8}z^8}$, &c. and extracting the root, $t=1+\frac{b^2}{2a^2}z^2-\frac{4b^3+b^4}{8a^4}z^4+\frac{8b^4+4b^5+b^6}{16a^6}z^6$, *&c.* And hence (by PROB. II.) may be had the length of the Hyperbolical Arch, or $t=z+\frac{b^2}{6a^2}z^3-\frac{4b^3+b^4}{40a^4}z^5+\frac{8b^4-4b^5+b^6}{112a^6}z^7$, *&c.*

If the Ellipsis $\sqrt{aa-bzz}=y$ were proposed; the sign of b ought to be every where changed, and there will be had $z+\frac{b^2}{6a^2}z^3+\frac{4b^3-b^4}{40a^4}z^5+\frac{8b^4-4b^5+b^6}{112a^6}z^7$, *&c.* for the length of its Arch; and likewise putting unity for b, it will be $z+\frac{z^3}{6a^2}+\frac{3z^5}{40a^4}+\frac{5z^7}{112a^6}$, *&c.* for the length of the circular Arch. Now the Numeral Co-efficients of this series may be found *ad infinitum*, by multiplying continually the terms of this Progression $\frac{1\times1}{2\times3}$, $\frac{3\times3}{4\times5}$, $\frac{5\times5}{6\times7}$, $\frac{7\times7}{8\times9}$, $\frac{9\times9}{10\times11}$, *&c.*

Ex. 9. Lastly let the Quadratrix VDE be proposed whose vertex is V, A being the center,



and AV the semi-diameter of the interior circle, to which it is adapted, and the angle VAE being a right angle. Now any right line AKD being drawn

drawn through A, cutting the circle in K, and the Quadratrix in D; and the perpendiculars KG, DB, being let fall to AE, call AV$=a$, AG$=z$, VK$=x$, and BD$=y$, and it will be as in the foregoing example $x=z+\frac{z^3}{6a^2}+\frac{3z^5}{40a^4}+\frac{5z^7}{112a^6}$, &c. Extract the root z, and there will arise $z=x-\frac{x^3}{6a^2}+\frac{x^5}{120a^4}-\frac{x^7}{5040a^6}$, &c. whose square subtract from $\overline{AK}^2$ or a^2, and the root of the remainder $a-\frac{x^2}{2a}+\frac{x^4}{24a^3}-\frac{x^6}{720a^5}$, &c. will be GK. Now whereas by the nature of the Quadratrix it is AB $=$VK$=x$; and since it is AG : GK :: AB : BD(y); divide AB×GK by AG, and there will arise $y=a-\frac{2x}{3a}-\frac{x^4}{45a^3}-\frac{2x^6}{945a^5}$, &c. And thence by PROB. I. $\dot{y}=-\frac{2x}{3a}-\frac{4x^3}{45a^3}-\frac{4x^5}{315a^5}$, &c. to the square of which add 1, and the root of the sum will be $1+\frac{2xx}{9a^2}-\frac{14x^5}{405a^4}+\frac{604x^6}{127575a^6}$, &c.$=\dot{t}$: whence (by PROB. I.) t may be obtained, or the arch of the Quadratrix, *viz.* VD $=x-\frac{2x^3}{27a^2}+\frac{14x^5}{2825a^4}+\frac{604x^7}{893025a^6}$, &c.

F I N I S.

ERRATA.

PAGE 2, *line* 18, *instead of* decuple, *read* decimal.

Page 40, *line* 9, *instead of* $\frac{\dot{y}}{\dot{x}}$ *read* $\frac{\dot{x}}{\dot{y}}$.

Page 43, *line* 3, *instead of* any aggregate, *read* an aggregate.
Page 94, *line* 3, *after* (AD), *insert* : : *k*K : *de*$=$*y*o, where I assume $x=1$ as above. Also CG : GF.
Page 101, *line* 10, *dele* BD.
Page 113, *line* 17, *instead of* AFD, *read* FDH.

Page 114, *line* 6, *after* $y=$, *read* $-\frac{e^3}{z^3}\sqrt{az-cc}$.

Page 140, *begin the paragraph numb.* 3. *thus.* The Series of every Order, except the second of the first Table, may be produced *in infinitum*. For in the Series of the third and fourth Order of the first Table the numeral Coefficients of the initial terms are formed by multiplying the numbers 2, —4, 16, —96, 768, &c. continually into each other; and the coefficients of the subsequent terms are derived from the initials in the third Order by multiplying gradually by $-\frac{1}{2}$, $-\frac{5}{4}$, $-\frac{7}{6}$, &c. or in the fourth Order by multiplying by $-\frac{1}{2}$, $-\frac{3}{4}$, $-\frac{5}{6}$. But the Coefficients of the denominators 1, 3, 15, 105, &c. arise by multiplying the numbers 1, 3, 5, 7, &c. gradually into each other.
Page 140, *line* 3 *from the bottom, instead of* fourth Order, *read* first Order.
Page 141, *line* 12, *instead of* the Series of the said fifth Order, *read* the Series of the third Order of the latter Table.
Page 173, *line* 10, *instead of pag.* 143. *read pag.* 169.

Mbl 1287869